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On the Fundamentals of Geometry

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Preface

The origin of this little book dates back to 1994, when I studied Pierre Samuel's *Projective Geometry* (see [Samuel]). The wonderful development of the affine and projective planes from very simple axioms, leading to profound results, appealed to me greatly. But the extension to higher dimensions was less satisfactory to my taste, and that stimulated me to go into more detail. Shortly afterwards I found in [Jacobson] a chapter on lattice theory, which appeared very suitable for describing projective spaces independent of dimension. This led to a first - unpublished - article in 1995. But one very awkward axiom was that of Dedekind: complicated and not at all intuitively acceptable in ordinary geometry. It left me very uneasy.
Some five years later I took up this project again, mainly to find alternatives for this Dedekind axiom. It appeared that, indeed, it was possible to replace it by two much more simple and intuitive axioms, and I rewrote the article for use in a small circle of colleagues. In 2006 I gave a lecture on the fundamentals of geometry, and in 2008 one on the equivalence of the theorem of Pappos with the commutativity of multiplication of scalars (see section 4.6). By then the article had grown beyond the size fit for a magazine. So in the summer of 2009 I decided to work out several details and make a little book of it.

And here it is. Apart from the great mathematicians who inspired me - like Samuel and Artin - I am indebted to many people who stimulated or helped me. First of all my wife, who tolerated my working on maths while eating her bread. Secondly my life-long friend Ruud Pellikaan, who led me out of confusion more than once, with his broad and profound knowledge. And thirdly the colleagues and students who patiently endured the sometimes alien world I tried to take them into.

This book is written for mathematicians, philosophers and theoretical physicists who want a sound fundament for geometry. It hardly contains any new facts. But it does give several new proofs and, above all, an intuitive way of developing the impressive cathedral of geometry from simple axioms that are immediately accepted as true in a real geometrical space. Now, if you glance through this book you might think that it is more about abstract algebra than about geometry. This is because geometry is in its essence algebra. But throughout we have tried to keep in touch with geometrical content. However, in order to make sure that there are no gaps in the proofs one has to ascend to the algebraic level of relations between geometrical objects.

There is another reason why this work may be of interest, apart from presenting yet another axiom system. In a time when geometry is reduced to calculus, if not completely absent, it is of utmost importance to keep the field alive. Even more so since in physics and computer science the interest in Clifford/Grassmann algebra is growing and old concepts like (linear) complexes are reappearing on the scene. So I also hope that this book modestly contributes to a better understanding and increased appreciation of true geometry.

Rotterdam, Autumn 2009
Lou de Boer

Contents

Chapter 1

Preliminaries

1.1 Prerequisites

The reader is assumed to be familiar with the basic facts of modern geometry, especially projective geometry, and, above all, with the gist of axiomatizing. In [Coxeter] one can find everything about geometry that is needed for this book, and much more. In addition she/he should have a basic knowledge of algebra and linear algebra: any undergraduate courses should be sufficient. It is not necessary to know about lattices, but in [Jacobson] you can find the most important things.

1.2 What is Geometry?

In ancient times, geometry was the study of physical space. It was used to predict the positions of the stars, to demarcate pieces of land, and to build houses and temples. It was concerned with points, lines, planes etc.

In Euclid's time it was converted from an experimental science into a deductive one, but there was no clear distinction between geometrical and physical space.

In the 19th century other kinds of geometry, viz. hyperbolic and elliptic ones, were discovered, and from then on a distinction had to be made between real physical space and all kinds of geometrical spaces. But still geometry was about points, lines and planes, and these had geometrical contents that were well agreed upon.

In the 20th century, with the rise of formal mathematics, these contents lost their importance. Though they still play an essential role in the development of mathematical consciousness of the individual, in academic circles one cherishes the abstractness of mathematical concepts and the freedom to substitute virtually everything into every 'undefined concept': e.g. we like to speak of a function being a point in a function space, or a complex number being a point in the Gauss plane.

Yet - certainly in physics - points, lines and planes still have their intuitive meaning. Thus, today we have to distinguish three kinds of space:

- physical space
- concrete geometrical spaces, in which points, lines and planes have the intuitive meaning of former times
- abstract mathematical spaces that are defined formally from certain axioms, and in which points need not be geometrical objects.

In what follows we will be concerned mostly with concrete geometrical spaces, though we will try to construct them with formal methods. And, of course, an important question still is: *which concrete geometrical space best fits our physical space?*

1.3 The Fundamentals of Geometry

In any formal mathematical theory one starts with undefined concepts and axioms. Then one develops new concepts and proves new facts.

It is hardly necessary to mention that before one has such an ideal theory, a long time of struggle may have passed, with vague concepts and wrong statements. As stated above, in Euclid's time the facts of geometry were converted into a deductive system. And at the end of the 19th century the whole mathematical landscape, including all kinds of geometry, was axiomized. So, founding a theory on clear concepts and axioms, is not done before the outlines of the theory have been developed to a certain extent.

In this treatise we will try to give a new system of axioms from which geometry can be developed. The reason why we think this is appropriate is formulated in section 1.6.

1.4 Elements

Before starting our theory, it is worthwhile to reflect on the nature of our elements: of points, lines and planes. A point is meant to be a 'locater', something that indicates a location in space. It has no size: no length, nor area, nor volume.

From 'point' one goes on to 'line': the (straight) line has infinite length, but no other sizes. It is infinitely thin. A good alternative to this is to look at the line as a border. In figure 1.1 there are two contrasting colours, black and white. A line is considered to be the border between these colours. As such the line is invisible, but very easy to imagine. It has a mental content, but no material one.

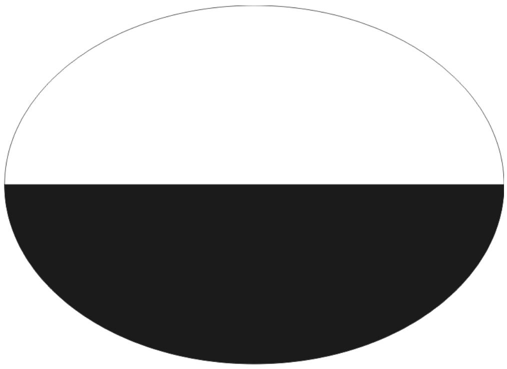

Figure 1.1: the line as a border

From this concept of line it is easy to get to 'point' and 'plane': as soon as two lines in space meet, there is a point as well as a plane.

The plane itself can also be thought of as the border between the water of a perfectly still lake and the air above it; that is, if, for the moment, we forget about molecules and stuff like that.

Looking back at our 'definition' of 'line', we note that we needed a plane. Without our two colours in the planar figure 1.1 it would be impossible to give this nice example. In Chapter 4 we will meet a very profound phenomenon (the construction of the vector space) that will make it clear that a geometrical line cannot exist on its own, but needs a plane.

Also we will see, in section 2.9, that the plane can hardly exist without a third dimension, to prove the Desargues statement.

So, in our treatise the 'dimension' of our spaces will be 3 or more, or at least we will suppose that our lines and planes can be embedded in a 3-dimensional space.

1.5 Projective Geometry

Geometry is about points, lines, curves, planes, surfaces, distances, angles... . But also about incidence, and incidence is more fundamental than distance or angle. Projective Geometry is about incidence, and not so much about distance and angle. Elliptic, Parabolic (=Euclidean) and Hyperbolic Geometry can all be derived from Projective Geometry by adding a suitable metric to (a subset of) it. In that sense Projective Geometry is more fundamental than the other geometries. So, if we want to investigate the fundamentals of Geometry, it is only natural to do this firstly and mainly on Projective Geometry.

1.6 Axiom systems

From a formal mathematical point of view, given two sets of consistent axioms, A and B, that are, in addition, equivalent, there is no reason to prefer one over the other. In fact the word 'prefer' has no formal mathematical meaning. Yet one can prefer A over B for didactical reasons, or because A is more beautiful than B, or A has fewer or simpler axioms than B, or because A is more 'intuitive' than B.

There are several ways to develop projective geometry. We list a few.

1. First we have the **historical** way in which it was 'discovered', starting from Euclidean geometry and adding elements at infinity; see e.g. [Reye] 2. Vortrag. Duality is not apparent and the underlying field is $\mathbf{R}$. By the way, every finite dimensional vector space over any field can be transformed into a projective space, by adding elements at infinity.

2. A second way - restricted mostly but not necessarily to the two and three dimensional cases - starts from points and lines (and planes) and incidence, the concept of dimension being immediate; see e.g. [Coxeter]. One could call it the **geometrical** approach. Duality is - in general - built in and the field of scalars is $\mathbf{R}$.

3. Since we know a great deal about fields and vector spaces, the most comfortable method for developing projective geometry is without doubt to start from a finite dimensional vector space and to identify dependent vectors; see e.g. [Samuel]. One could call it the **numeric** approach, since from the very beginning all points are fully determined by tuples of numbers. Working with lines, planes etc. is, however, much less easy. Duality is not immediate, but can be achieved via the dual vector space, i.e. with linear functions. Any field is allowed.

4. Another one, see [Aigner] page 55, starts with a set of points and an elected collection of subsets, the lines. A higher dimensional subspace is constructed inductively from a lower dimensional one and one point

outside it. This could be called the **set-theoretical** approach. Duality is not apparent but the underlying field can be chosen arbitrarily.

5. In [Stoss] real 3-dimensional projective geometry is developed in a very elegant way, starting from 'line' and 'touch' (Treffen) only. Duality is built in, the field is **R**.

6. A sixth one, which is presented here, starts from lattice-theory, and hence is essentially **algebraic**. Our affinity towards it is because

- the few and simple axioms of lattice theory are immediately accepted to be true in our 'real projective' world; in that sense this approach appears to be highly *phenomenological* or *intuitive*
- it is dimension-independent
- here - like in several other courses, but by no means all - our projective space is not considered to be a set of points only; treating lines and planes as objects on their own rather than as point-sets has the disadvantage that the results of set-theory cannot be used that much, but *duality* becomes explicit and clear
- the somewhat wide concept of 'incidence' is replaced by the more narrow one of 'containing', thus turning our space into a partially ordered set
- the rather tiring and cumbersome axioms of separation ([Coxeter] Chapter II) become easy theorems after choosing **R** as our ground field (this is not presented in this book, however)
- in many textbooks on Projective Geometry duality is formulated in terms of point, line, plane and incidence; the dual counterpart of the whole space - the empty set - I found nowhere explicitly mentioned, though they form perfect initial (smallest) and terminal (biggest) elements, that is zero and unity; in lattice theory they are basic dual concepts
- expressions like 'the plane determined by a line l and a point p' or 'the line through the points p and q' are reformulated in terms of *join*: $l \vee p$ resp. $p \vee q$, thus covering all dimensions; expressions like 'the intersection of lines l and m' or 'the meeting point of line l and plane α' are reformulated by *meet*: $l \wedge m$ resp. $l \wedge \alpha$; it has the additional advantage that one can define polynomials and with them *degree of freedom* as an alternative for the awkward concept of ∞^n; which - by the way - is not presented in this treatise
- finally, the classical geometries, like the point range, the flat pencil, point-geometry etc., appear in a uniform way as intervals.

1.7 The structure of this book

In the next chapter we will present our Axiom System. The main question then is to prove that our spaces are projective spaces in the sense of one of the other systems. That requires the concept of isomorphic spaces. Therefore, in Chapter 3 we will be concerned with maps between projective spaces. In Chapter 4 we will show that indeed our system is equivalent to the other ones by proving that our space is isomorphic to the lattice of subspaces of a vector space.

As usual $\mathbf{Z}$, $\mathbf{Q}$ and $\mathbf{R}$ denote the sets of integers, rationals and reals respectively; $\mathbf{F}_p$ is the finite field with p elements. A small circle, $\circ$, denotes the composition of maps; a dagger, $\dagger$, means a contradiction; a 'diamond', $\diamond$, means the end of an example or exercise or proof, or sometimes 'trivial proof'.

'Skew field' is the same as 'division ring', meaning a ring in which the non-zero elements form a multiplicative group. A *field* is a division ring in which multiplication is commutative. Throughout this book 'vector space' is meant to include modules over skew fields. In particular 'vector space' means 'left vector space' if the skew field is not commutative.

Chapter 2

The Axiom System

The first part of this chapter, sections 2.1 to 2.4 is a paraphrase of Jacobson's lattice theory (see [Jacobson] Chapter VII) applied to geometry. In sections 2.7 to 2.9 we add axioms to turn these lattices into projective spaces.

2.1 Size

As stated before, geometry is about points, lines and planes. So the first thing to notice is the existence of different *types* of elements. These types are finite in number, but not unrelated like 'apple' and 'democracy': we think of a point as being 'smaller' than a line, which in turn is smaller than a plane. This means that we can order the types, i.e. we can associate integers to them. Points are agreed to have size or *dimension* 0, lines 1 and so on.

So, in developing our formal theory we will start with

- a set S, our *space*,
- an integer $n \geq 3$ and
- a surjective function *dimension*

$$\dim : S \rightarrow \{-1, 0, \ldots n-1, n\}$$

Note that for every integer k between -1 and n there is at least one element $x \in S$ with $\dim(x) = k$.

The condition $n \geq 3$ needs some explanation. First, $n \geq 1$ guarantees that S is not empty, or rather, as will be shown, that it has at least

one line containing at least one point. As a consequence, we deliberately exclude 0-dimensional spaces. The reason for this you will find on page 25 in the footnote.
By taking $n \geq 2$ we allow the constructions in Chapter 4, without which we are left without any number system. If $n = 2$ we would need the not very intuitive proposition of Desargues (2.9.8) as an axiom. See [Samuel] 1.4 for a description of a non-desarguian plane. You will agree that this weird example does not satisfy our common image of geometry. Of course we will not exclude 1- and 2-dimensional spaces, but we will assume that these can be embedded in a 3-dimensional one. By experience we know we live in a 3-dimensional world, so this is certainly not contrary to our intuition.

Definition 2.1.1 *The number n is called the* dimension *of S.*

Since we want to develop our theory not only for the projective plane and space, but also, for example, for the 5-dimensional space of linear complexes, we do not restrict to $n = 3$. However, our geometrical examples are taken from projective 3-space, unless explicitly mentioned otherwise.

Definition 2.1.2 *For $x \in S$:*
if $\dim(x) = 0$ *then x is called a* point*,*
if $\dim(x) = 1$ *it is called a* line*, and*
if $\dim(x) = 2$ *it is called a* plane.
In general, an element x is called a k-blade *or* linear k-manifold *if* $\dim(x) = k$.

In section 2.3 we will see what $\dim(x) = -1$ means.

Instead of our function 'dim' one could take the function

$$p + \dim : S \to \{p - 1, p, \ldots n + p - 1, n + p\}$$

for any integer p. In the literature one particularly finds the function rank$= 1 + \dim$.

In section 2.5 we will focus on the symmetry of our space which is known as the *principle of duality*. To establish this symmetry from the very beginning, we introduce the - not very useful - concept of 'co-dimension'.

Definition 2.1.3 *The* co-dimension *of an element $x \in S$ is the number*

$$\mathrm{codim}(x) = n - 1 - \dim(x)$$

Note that

$$\mathrm{codim} : S \to \{-1, \ldots n\}$$

and that it is surjective.

Definition 2.1.4 *For $x \in S$:*
if $\mathrm{codim}(x) = 0$ *then x is called a* dual point *or a* hyperplane*,*
if $\mathrm{codim}(x) = 1$ *it is called a* dual line*,*
if $\mathrm{codim}(x) = 2$ *it is called a* dual plane*.*

Corollary 2.1.5 *Dual points have dimension $n-1$, dual lines $n-2$ and dual planes $n-3$.* $\diamond$

2.2 Order

The next thing to observe is that the line is not only bigger than the point, but it also *contains* points, and a plane contains lines and points. A point can be *in* - or, if you like: *on* - a line or plane, and a line can lie in a plane. We apparently have a binary relation I on S that we want to satisfy the following axiom.

Axiom 2.2.1 of order
There is a binary relation $I \subset S \times S$ which for every $x, y, z \in S$ satisfies:

$(x, x) \in I$, *i.e. the relation is* reflexive

$(\ (x, y) \in I \ \ and\ \ (y, x) \in I\ ) \ \Rightarrow\ x = y,$
i.e. the relation is anti-symmetric

$(\ (x, y) \in I \ \ and\ \ (y, z) \in I\ ) \ \Rightarrow\ (x, z) \in I,$
i.e. the relation is transitive

This axiom states that our space S is a *partially ordered* set.

Definition 2.2.2 *The following sentences all mean $(x, y) \in I$:*

x is in y, *or* x is contained in y, *notation* $x \preceq y$

y contains x, *or y passes/goes through*[1] x, *notation* $y \succeq x$

Proposition 2.2.3 *For every $x, y \in S$:*

$$(x \preceq y \ and\ y \preceq x) \ \Leftrightarrow\ x = y$$

Proof. $\Rightarrow$ is the axiom. Now if $x = y$ then from reflexivity $x \preceq y$ and $y \preceq x$ follow. $\diamond$

If two elements are *not* equal we use the following symbol.

[1] In a similar way we say that a road passes/goes through some city. This suggests a movement, but in fact it is a variable point X 'moving' on l that passes through P.

Definition 2.2.4 *For every $x, y \in S$:*

$$x \prec y \quad \textit{and} \quad y \succ x \quad \textit{both mean:} \quad (x \preceq y \textit{ and } x \neq y)$$

The elements x and y are called *comparable* if $x \preceq y$ or $y \preceq x$. In the real numbers we have either $x < y$ or $y > x$ or $x = y$, for any numbers x and y, that is: all numbers are comparable. The reals are *linearly ordered*. In our space S two elements may be uncomparable: if P is a point not on a line l, neither $P \prec l$, nor $P \succ l$, nor $P = l$ holds. Our space is not linearly ordered.

Definition 2.2.5 *For every $x, y \in S$:*

$$x \not\preceq y \quad \textit{and} \quad y \not\succeq x \quad \textit{both mean:} \quad (x, y) \notin I$$

Similarly $x \not\prec y$ and $y \not\succ x$ are the negations of $x \prec y$.

Note that $x \not\prec y$ holds if $x = y$. But we will avoid using $\not\prec$ and $\not\succ$.

For future use we have the following trivial property.

Proposition 2.2.6 *If for every $x \in S$ holds $(x \preceq a \Rightarrow x \preceq b)$ then $a \preceq b$.*

Proof. Take $x = a$. $\diamond$

The ordering relation is stronger than dimension. In fact we want:

Axiom 2.2.7 of monotone dimension
For every $x, y \in S$ we have:

$$x \prec y \Rightarrow \dim(x) < \dim(y)$$

i.e. dim *is* strictly monotone.

Clearly the function *codim* is strictly monotone too, but 'descending'.

A trivial but useful application is the following

Proposition 2.2.8 *If $a \preceq b$ and $\dim(a) = \dim(b)$ then $a = b$.*

Proof. $a \preceq b$ means $a \prec b$ or $a = b$. In the former case we have $\dim(a) < \dim(b)$ †, which leaves the latter. $\diamond$

2.3 Borders

We generally accept that the points and lines of 2-dimensional geometry all lie in a plane, and that the points, lines and planes of 3-dimensional geometry all lie in space. That means, we accept the existence of a biggest element $\mathbf{P}_n$ that contains all elements of S. Less usual, but equally necessary for a symmetric construction of our theory, is the existence of a smallest element, viz. the empty set $\emptyset$, that is contained in every element of S.

Axiom 2.3.1 of border
There is an element $\mathbf{0}$ *such that for every* $x \in S : \mathbf{0} \preceq x$. *There is an element* $\mathbf{1}$ *such that for every* $x \in S : \mathbf{1} \succeq x$.

We will use $\mathbf{0}$ and $\mathbf{1}$ mostly in our general theory, and $\emptyset$ and $\mathbf{P}_n$ in geometric applications. However, as soon as we consider subspaces or intervals, e.g. the geometry of a point, $\mathbf{0}$ and $\mathbf{1}$ will get different interpretations, see section 2.6.

Proposition 2.3.2 *The element* $\mathbf{0}$ *is unique, i.e. if there is a* $p \in S$ *such that* $p \preceq x$ *for every* $x \in S$, *then* $p = \mathbf{0}$.

Proof. Suppose $p \preceq x$ for every $x \in S$, then in particular $p \preceq \mathbf{0}$. But by the axiom of border (2.3.1) also $\mathbf{0} \preceq p$.
Now by axiom 2.2.1 of anti-symmetry $p = \mathbf{0}$. $\diamond$

Proposition 2.3.3 *The element* $\mathbf{1}$ *is unique.* $\diamond$

Proposition 2.3.4 *For every* $x \in S$:

$$\dim(x) = -1 \Leftrightarrow x = \mathbf{0}$$

$$\dim(x) = n \;\Leftrightarrow\; x = \mathbf{1}$$

where n *is again the maximum value of the dimension function, see section 2.1.*

Proof. of the first proposition. Suppose $\dim(x) = -1$. Since $\mathbf{0} \preceq x$, either $\mathbf{0} = x$ or $\mathbf{0} \prec x$. In the latter case we have $\dim(\mathbf{0}) < \dim(x) = -1$ †; which leaves $x = \mathbf{0}$. Conversely, suppose $x = \mathbf{0}$. Then for every $y \neq \mathbf{0}$ we have $\mathbf{0} \prec y$, hence $\dim(\mathbf{0}) < \dim(y)$. Hence $\dim(\mathbf{0})$ is the smallest element of $\dim(S)$, viz. -1. Second proposition analogous. $\diamond$

Corollary 2.3.5 $\mathbf{0} \neq \mathbf{1}$

2.4 Meet and join

The basic techniques of geometry are meet and join. The join of two different points is a line, and the meet of a line and a plane is the meeting point, unless the plane contains the line. What exactly do we mean by meet and join?

Definition 2.4.1 *An element $p \in S$ is called an* upper bound *for a subset V of S if p contains each element of V: $x \in V \;\Rightarrow\; x \preceq p$.*
An element $p \in S$ is called a lower bound *for V if it is contained in each element of V: $x \in V \;\Rightarrow\; p \preceq x$.*
An element $p \in S$ is called a least upper bound *for a subset V of S if it is an upper bound for V and if it is contained in every other upper bound for V.*
An element $p \in S$ is called a greatest lower bound *for a subset V of S if it is a lower bound for V and if it contains every other lower bound for V.*

Note. Since $p \preceq p$ the word 'other' could be omitted.

Example. Let l be the meeting line of the planes α and β. Every point on l is a lower bound of the set $\{\alpha, \beta\}$ and l is its greatest lower bound. $\diamond$

Proposition 2.4.2 *Each subset V of S has at most one least upper bound and at most one greatest lower bound.*

Proof. Let a and b both be least upper bounds for V. Then $a \preceq b$ and $b \preceq a$, so $a = b$. Analogously for the greatest lower bound. $\diamond$

It is clear from axiom 2.3.1 that every subset of S has an upper and a lower bound. A least upper bound and greatest lower bound do not always exist in partially ordered sets. This can be seen from figure 2.1: an arrow in this and future figures means $\preceq$, so e.g. $a \preceq q$ and - from transitivity - $a \preceq 1$.

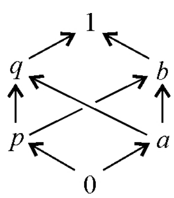

Figure 2.1: Non-unique bounds

In geometry this would mean, for instance, that two different lines can have two different common points. To exclude these situations we need the following axiom.

Axiom 2.4.3 Lattice axiom
Each pair of elements of S has a least upper bound and a greatest lower bound.

This axiom together with the axiom of order, 2.2.1, makes our space a *lattice*. It is a very strong axiom indeed. It implies, for instance, that two different points determine one line.

Definition 2.4.4 *The least upper bound of x and y is called their* join*, denoted by $x \vee y$. The greatest lower bound is called their* meet*, denoted by $x \wedge y$.*

Proposition 2.4.5 *For every $a, x \in S$: $a \wedge x \preceq a \preceq a \vee x$.*

Proof. Immediate from the definition of *greatest lower bound* and *least upper bound.* $\diamond$

Proposition 2.4.6 *For every $a, b, x \in S$: $(x \preceq a$ and $x \preceq b) \;\Rightarrow\; x \preceq a \wedge b$, and also $(x \succeq a$ and $x \succeq b) \;\Rightarrow\; x \succeq (a \vee b)$.*

Proof. Immediate from the definition of *greatest lower bound* and *least upper bound.* $\diamond$

Proposition 2.4.7 *Meet and join are commutative and associative, i.e. for every $a, b, c \in S$:*

$$\begin{array}{cc} a \wedge b = b \wedge a & a \vee b = b \vee a \\ (a \wedge b) \wedge c = a \wedge (b \wedge c) & (a \vee b) \vee c = a \vee (b \vee c) \end{array}$$

Proof. of $(a \wedge b) \wedge c = a \wedge (b \wedge c)$.
$((a \wedge b) \wedge c) \preceq (a \wedge b) \preceq a$, and $((a \wedge b) \wedge c) \preceq (a \wedge b) \preceq b$, and $((a \wedge b) \wedge c) \preceq c$, so $(a \wedge b) \wedge c$ is lower bound of $\{a, b, c\}$, hence of $a, b \wedge c$. So $((a \wedge b) \wedge c) \preceq (a \wedge (b \wedge c))$. Similar $((a \wedge b) \wedge c) \succeq (a \wedge (b \wedge c))$. $\diamond$

From this it follows that we can speak of *the* meet and join of any finite subset of S.

Proposition 2.4.8 *For every $a \in S$:*

$$\begin{array}{ll} \mathbf{0} \wedge a = \mathbf{0} & \mathbf{0} \vee a = a \\ \mathbf{1} \wedge a = a & \mathbf{1} \vee a = \mathbf{1} \quad \diamond \end{array}$$

Proposition 2.4.9 *For every* $a \in S : a \wedge a = a \vee a = a$. $\diamond$

That is: every element is *idempotent* with respect to both operations.

Proposition 2.4.10 *For every* $a, x \in S$*:*

$$(a \vee x) \wedge a = a \text{ and } (a \wedge x) \vee a = a$$

Proof. $a \preceq a$ and $a \preceq a \vee x$, hence $a \preceq (a \wedge (a \vee x))$. Conversely $(a \wedge y) \preceq a$ for all y, so - taking $y = a \vee x$ - also $(a \wedge (a \vee x)) \preceq a$. $\diamond$

Proposition 2.4.11 *For every* $a, b \in S$*:* $a \wedge b = a \Leftrightarrow a \preceq b \Leftrightarrow a \vee b = b$. $\diamond$

Proposition 2.4.12 *For every* $a, b, x \in S$*:*

$$\text{if } a \preceq b \text{ then: } (a \wedge x \preceq b \wedge x) \text{ and } (a \vee x \preceq b \vee x)$$

i.e. meet and join are monotone.

Proof. $a \wedge x \preceq a \preceq b$ and $a \wedge x \preceq x$ imply $a \wedge x \preceq b \wedge x$. $\diamond$

Corollary 2.4.13

$$(a \preceq b \text{ and } b \wedge x = \mathbf{0}) \implies a \wedge x = \mathbf{0}$$

and

$$(a \preceq b \text{ and } a \vee x = \mathbf{1}) \implies b \vee x = \mathbf{1} \diamond$$

We close this section with a trivial but useful fact.

Proposition 2.4.14 *If* $p \preceq a$ *but* $p \not\preceq a \wedge b$ *then* $p \not\preceq b$*. And dually: if* $a \preceq q$ *but* $a \vee b \not\preceq q$ *then* $b \not\preceq q$*.*

Proof. Logical consequence of proposition 2.4.6. $\diamond$

2.5 Duality

Note the *duality* between meet and join. The same duality exists between $\prec$ and $\succ$, between $\preceq$ and $\succeq$, and between $\mathbf{0}$ and $\mathbf{1}$. Suppose P is a statement involving variables over S, and some or all of the algebraic symbols $\vee \wedge \prec \succ \preceq \succeq$ dim codim $\mathbf{0}$ $\mathbf{1}$, and possibly brackets, logical symbols and numbers, but nothing else. The dual P^* of P is obtained by replacing each of those algebraic symbols of P by its dual. Then P^* is true if and only if P is true.

This meta theorem follows from the symmetric way we have constructed our theory so far.

Instead of working with the relation I in section 2.1, we could have taken the relation J defined by $(a,b) \in J \Leftrightarrow (b,a) \in I$. It will be clear that this leads to a completely similar structure, in which the order relations are reversed, and meet and join interchanged. This space will be called the *dual* space S^* of S.
In S^* the smallest element is $\mathbf{1}$ and the biggest $\mathbf{0}$.

2.6 Intervals

A well known fact from projective geometry is that a flat pencil of lines behaves very much the same as a range of points on a line, or as a pencil of planes in space. Likewise, a bundle of lines and planes is 'isomorphic' to the projective plane of points and lines. To formalize these concepts we need the following definition.

Definition 2.6.1 *The* open interval $\langle a,b\rangle$ *is the set of elements 'between' a and b:*

$$\langle a,b\rangle = \{x \in S | a \prec x \prec b\}$$

The closed interval $[a,b]$ *is defined as:*

$$[a,b] = \{x \in S | a \preceq x \preceq b\}$$

So if P is a point in a plane α, the interval $\langle P,\alpha\rangle$ is the flat pencil of lines through P in α. $\langle \emptyset, l\rangle$ is the range of points on line l, $\langle l, \mathbf{P}_3\rangle$ is the pencil of planes through l.

We leave the definition of a half-open interval to the reader.

Proposition 2.6.2 $[\mathbf{0},\mathbf{1}] = S$. $\diamond$

Proposition 2.6.3 *If a is not contained in b, then $[a,b]$ is empty.* $\diamond$

Proposition 2.6.4 *Suppose $a \prec b$ and $\dim(b) > \dim(a) + 3$. Define $\dim'(x) = \dim(x) - \dim(a) - 1$. Then $[a,b]$ satisfies all previous axioms with $\dim'$ instead of $\dim$ and with a as smallest and b as biggest element.*

The proof is left as an easy exercise for the reader.

Definition 2.6.5 *The number*

$$\dim([a,b]) = \dim(b) - \dim(a) - 1 = \dim'(b)$$

is called the dimension *of the interval* $[a,b]$*, provided that* $a \prec b$*. If the interval is empty or if* $a = b$*, its dimension is not defined.*

This is in accordance with definition 2.1.1.

Definition 2.6.6 *Closed intervals of dimension 1 or higher are called* subspaces *of* S*.*

Note that we allow subspaces to have dimension 1 or 2, whereas S has to be of dimension 3 or more. The reason for this will become clear in the sequel.

Note also that if $a \preceq b$, $[a,b]^*$ is the dual space of $[a,b]$. This has hardly any meaning if the dimension of the interval is 1.

2.7 Distributivity

Meet and join have a certain similarity with the AND and OR operators in predicate calculus. However, there is a big difference too: *distributivity* does *not* hold in each lattice. In general $a \wedge (b \vee c) \neq (a \wedge b) \vee (a \wedge c)$ and $a \vee (b \wedge c) \neq (a \vee b) \wedge (a \vee c)$.

Example. Let a and d be different lines with common point E, and B and C different points on d but not on a, see figure 2.2. Then $a \wedge (B \vee C) = a \wedge d = E$ and $(a \wedge B) \vee (a \wedge C) = \mathbf{0} \vee \mathbf{0} = \mathbf{0}$. Also $a \vee (B \wedge C) = a$ and $(a \vee B) \wedge (a \vee C) = a \vee d$, being a plane. $\diamond$

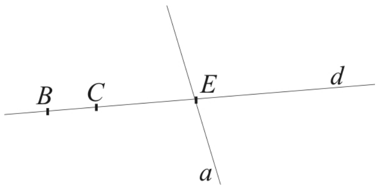

Figure 2.2: No distributivity

We do, however, have a weaker property.

Proposition 2.7.1 *For every* $a, b, c \in S$*:*

$$(a \wedge b) \vee (a \wedge c) \preceq (a \wedge (b \vee c))$$

and

$$(a \vee b) \wedge (a \vee c) \succeq a \vee (b \wedge c)$$

Proof. $b \preceq b \vee c$ so $(a \wedge b) \preceq (a \wedge (b \vee c))$; similar $a \wedge c \preceq a \wedge (b \vee c)$. $\diamond$

As a direct consequence of this proposition we have the following one, a bit odd, at first sight, yet very important.

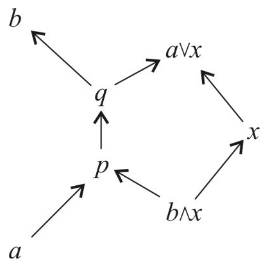

Figure 2.3: proposition 2.7.2

Proposition 2.7.2 *For every $a \preceq b$ and arbitrary x: $a \vee (b \wedge x) \preceq b \wedge (a \vee x)$.*

Proof. Put $p = a \vee (b \wedge x)$ and $q = b \wedge (a \vee x)$, see figure 2.3. From proposition 2.7.1 we have $p = a \vee (b \wedge x) \preceq (a \vee b) \wedge (a \vee x) = b \wedge (a \vee x) = q$. $\diamond$

In many lattices, including projective spaces, we have the stronger $a \vee (b \wedge x) = b \wedge (a \vee x)$ for $a \preceq b$. But that does not hold in every lattice. As an example, consider the lattice from figure 2.4. In this lattice we have $u \prec v$ and $p = u \vee (v \wedge s) = u$ and $q = v \wedge (u \vee s) = v$, hence $p \prec q$. Note that the lattice of figure 2.4 is part of the lattice of figure 2.3.

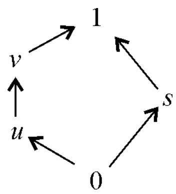

Figure 2.4: A non-modular lattice

Lattices that *do* satisfy the property $p = q$ are called *modular* or *Dedekind*.

In figure 2.5 we indicate that this property may hold in real 3-dimensional space: A is a point in a plane β and x a line that meets β in $P \neq A$.

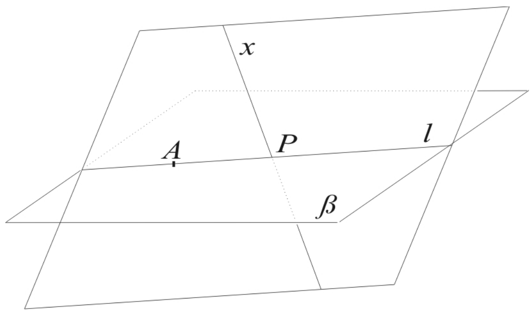

Figure 2.5: Modularity in geometry

Then $A \vee (\beta \wedge x) = \beta \wedge (A \vee x) = l$.

To turn our space into a modular lattice, we need, however, the additional axioms of the next section.

2.8 The dimension theorem

Example. In real projective 3-space, let α be a plane, l a line that meets the plane in a point P. Then:

$$\dim(\alpha) + \dim(l) = 2 + 1 = 3$$

$$\dim(\alpha \wedge l) + \dim(\alpha \vee l) = \dim(P) + \dim(\mathbf{P}_3) = 0 + 3 = 3 \diamond$$

The reader is invited to verify that in real projective space for all a, b holds:

$$\dim(a) + \dim(b) = \dim(a \wedge b) + \dim(a \vee b)$$

To prove this central theorem in our theory, two more axioms are needed. The first requires the existence of 'enough' points to construct arbitrary elements of our space. The second guarantees that the space has sufficient symmetry: joining any line with a point not on it, will invariably give a plane, etc., see figure 2.6.

Axiom 2.8.1 of sufficient points/hyperplanes
For every pair a, b of elements of S for which $a \prec b$, there exists a point x, i.e. $\dim(x) = 0$, *such that*

- $x \not\preceq a$
- $x \preceq b$

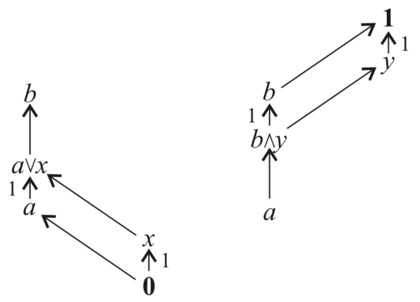

Figure 2.6: Composition

And dually, after translation: for every pair a, b of elements of S for which $a \prec b$, there exists a hyperplane y, i.e. $\dim(y) = n - 1$*, such that*

- $a \preceq y$
- $b \not\preceq y$

Axiom 2.8.2 of composition
If x is a point and a any element not containing x, then $\dim(a \vee x) = \dim(a) + 1$*. And dually: if y is a hyperplane and b any element not in y, then* $\dim(b \wedge y) = \dim(b) - 1$*.*

As an immediate consequence of the first axiom we have the following fundamental property; compare proposition 2.2.6.

Proposition 2.8.3 *If for every point x holds $(x \preceq a \Rightarrow x \preceq b)$, then $a \preceq b$. And dually: if for every hyperplane y holds $(b \preceq y \Rightarrow a \preceq y)$, then $a \preceq b$.*

Proof. (First statement only.) Suppose $a \wedge b \prec a$, then by the previous axiom there must be a point p such that $p \preceq a$ and $p \not\preceq a \wedge b$. By hypothesis we then have $p \preceq b$, hence $p \preceq a \wedge b$ †. That means our supposition is false, i.e. $a \wedge b = a$, i.e. $a \preceq b$. $\diamond$

Definition 2.8.4 *Let i, k be integers,* $0 < i \leq k$*.*
A tuple $(a_0, a_1 \ldots a_k)$ is called a chain of length k *if for each i :* $a_{i-1} \preceq a_i$*.*
The chain is irredundant *if $a_{i-1} \prec a_i$ for all i, otherwise it is* redundant*.*
The chain is called complete *if for each $x \in S$ and for each i:*

$$a_{i-1} \preceq x \preceq a_i \Rightarrow (a_{i-1} = x \text{ or } x = a_i)$$

A complete chain $(a_0, a_1 \ldots a_k)$ is said to connect a_0 *and* a_k*.*
A composition chain *is a complete and irredundant chain.*

Examples. Let P, Q be different points on a line l, and this line in a plane α. Then

- (P, Q, α) is not a chain, neither are (l, P, α) and $(Q, \emptyset)$.
- (P, P, α) is a chain of length 2, neither irredundant nor complete
- $(\emptyset, Q, l, l)$ is a complete and redundant chain of length 3 connecting $\emptyset$ and l
- $(\emptyset, l, \mathbf{P}_3)$ is an irredundant but not complete chain of length 2
- $(P, l, \alpha, \mathbf{P}_3)$ is a composition chain of length 3 connecting P and $\mathbf{P}_3$.$\diamond$

Proposition 2.8.5 *If* $(a_0, \ldots a_k)$ *is a composition chain, then* $\dim(a_i) = \dim(a_{i-1}) + 1$ *for each* $i = 1 \ldots k$.

Proof. If $(a_0, \ldots a_k)$ is a composition chain, $a_{i-1} \prec a_i$. Then from axiom 2.8.1 we know that there is a point x_i such that $x_i \not\preceq a_{i-1}$ and $x_i \prec a_i$. Let $y_i = a_{i-1} \vee x_i$, then $a_{i-1} \prec y_i \preceq a_i$. Because the chain is complete, $y_i = a_i$. Furthermore, according to axiom 2.8.2: $\dim(a_i) = \dim(y_i) = \dim(a_{i-1}) + 1$. $\diamond$

Proposition 2.8.6 *If* $a \prec b$ *there exists a composition chain connecting them. Any composition chain connecting* a *and* b *has length* $\dim(b) - \dim(a)$.

Proof. From axiom 2.8.1 we know that there is a point $x_1 \preceq b$, $x_1 \not\preceq a$. Let $y_1 = a \vee x_1$. Then $a = y_0 \prec y_1 \preceq b$. If $y_1 = b$ we have a composition chain, of length 1. Note that by axiom 2.8.2 $\dim(y_0) + 1 = \dim(y_1)$. If $y_1 \neq b$ we can repeat this process. Suppose we have constructed composition chain $(y_0, \ldots y_{k-1})$ with $y_{k-1} \prec b$. Then there must be a point $x_k \preceq b$, $x_k \not\preceq y_{k-1}$. Define $y_k = y_{k-1} \vee x_k$. Now we have $\dim(y_{k-1}) + 1 = \dim(y_k)$, hence $\dim(a) + k = \dim(y_k)$. As soon as $\dim(y_k) = \dim(b)$ the process stops, and $y_k = b$.
Now let $(a = y_0, \ldots y_k = b)$ and $(a = z_0, \ldots z_l = b)$ be composition chains connecting a and b. Iterating proposition 2.8.5 we find that $\dim(b) = \dim(y_k) = \dim(y_0) + k$. In the same way we find $\dim(b) = \dim(z_k) = \dim(z_0) + l$, whence $k = l = \dim(b) - \dim(a)$. $\diamond$

Proposition 2.8.7 *Let* $b \not\preceq a$, $a \wedge b \prec z \preceq b$ *and*

$$\dim(z) = \dim(a \wedge b) + 1$$

Then $(a \vee z) \wedge b = z$.

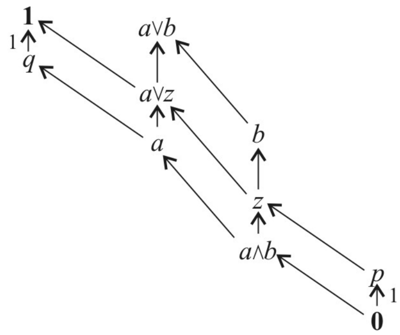

Figure 2.7: Proposition 2.8.7

Proof. From the hypothesis it follows by the last two axioms that there exists a point p such that $p \prec z,\ p \not\preceq a \wedge b,\ z = (a \wedge b) \vee p$, see figure 2.7. Now, since $p \not\preceq a \wedge b$ and $p \prec z \preceq b$, p can't be in a, see proposition 2.4.14. Also $a \vee z = a \vee (a \wedge b) \vee p = a \vee p$, hence $\dim(a \vee z) = \dim(a) + 1$. So $a \prec a \vee z$. Then there must exist a hyperplane q such that and $a \preceq q,\ a \vee z \not\preceq q$ and $\dim(q \wedge (a \vee z)) = \dim(a \vee z) - 1 = \dim(a)$. Since $a \preceq q$ and $a \preceq a \vee z$ we have $a \preceq q \wedge (a \vee z)$ so $a = q \wedge (a \vee z) = q \wedge (a \vee p)$. Next we meet $a \vee z$ with b. Put $z' = (a \vee z) \wedge b$. We have to prove that $z = z'$. We know from 2.7.1 that $z' = b \wedge (a \vee z) \succeq (b \wedge a) \vee (b \wedge z) = z$. From $(a \vee p) \wedge q = a$ follows

$$(a \vee p) \wedge q \wedge b = a \wedge b$$

$$z' \wedge q = a \wedge b$$

Is it possible that $z' \preceq q$? That would mean from the last equation: $z' = a \wedge b$. But that is not possible since $a \wedge b \prec z \preceq z'$. Now we can apply our axiom 2.8.2 again: $\dim(z') = \dim(z' \wedge q) + 1 = \dim(a \wedge b) + 1 = \dim(z)$, i.e. $z = z'$. $\diamond$

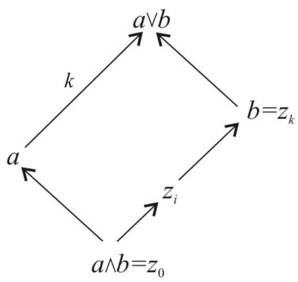

Figure 2.8: Proposition 2.8.8

Proposition 2.8.8 *Let* $b \not\preceq a$, $a \wedge b \prec z \preceq b$. *Then* $(a \vee z) \wedge b = z$.

Proof. Let $k = \dim(z) - \dim(a \wedge b)$. For $k = 1$ we have the previous theorem. Now suppose the statement correct for $k = m$ and suppose $\dim(b) - \dim(a \wedge b) > m$. Take any composition chain $(z_0 = a \wedge b, \ldots, z_m, z = z_{m+1}, \ldots z_j = b$. Then by assumption we have $z_m = (a \vee z_m) \wedge b$. Also $a \vee b = (a \vee z_m) \vee b$. Hence we can apply the previous theorem on the elements $a \vee z_m$, b and z, from which the statement for $k = m + 1$ follows. $\diamond$

As a corollary we have our central dimension theorem.

Proposition 2.8.9 Dimension theorem
For any two elements $a, b \in S$:

$$\dim(a) + \dim(b) = \dim(a \wedge b) + \dim(a \vee b)$$

Proof. In theorem 2.8.8 take $z = b$. Then $a \vee z = a \vee b$ and the chains $(z_0 = a \wedge b, \ldots, z_k = b)$ and $(a = a \vee z_0, \ldots, a \vee z_k = a \vee b)$ have the same length k. By definition the first chain is a composition chain, and from the proof of theorem 2.8.8 follows immediately that the second is a composition chain too. Now we have $\dim(a \vee b) - \dim(a) = k = \dim(b) - \dim(a \wedge b)$, from which the dimension theorem follows. $\diamond$

Now we are also in the position to prove that our space is modular.

Proposition 2.8.10 *For every* $a, b, x \in S$: $a \preceq b \;\Rightarrow\; a \vee (b \wedge x) = b \wedge (a \vee x)$.

Proof. See figure 2.3. Put again $p = a \vee (b \wedge x)$ and $q = b \wedge (a \vee x)$.
If $a \preceq x$ one easily verifies that $p = q = b \wedge x$, and if $x \preceq b$ we have $p = q = a \vee x$.
So, suppose $a \not\preceq x \not\preceq b$. Then we have $b \wedge x \preceq p \preceq b$. From proposition 2.8.8 follows $(p \vee x) \wedge b = p$ or $(a \vee (b \wedge x) \vee x) \wedge b = p$ or $(a \vee x) \wedge b = p$ or $a = p$. $\diamond$

This proof we owe to M. Aigner, see [Aigner], page 43.

2.9 Projective spaces

We will now proceed to the definition of a projective space. In order to exclude too simple spaces, like the line with two and the plane with three points[2] we need the following axiom.

[2]These have no vector space structure over a field.

Axiom 2.9.1 of cardinality
Every line has at least three points on it.
Dually: every dual line is contained in at least three dual points.

In our 'real' geometry we will deal with lines with infinitely many points, see considerations in section 4.9. But at this stage such a restriction is not necessary.

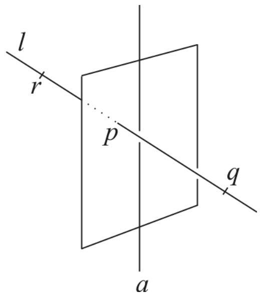

Figure 2.9: 1-dimensional intervals

Proposition 2.9.2 *Let* $a \prec c$ *and* $\dim(c) = 2 + \dim(a)$*. Then there are at least three elements* b_1, b_2, b_3 *with* $a \prec b_i \prec c$ *(and hence* $\dim(b_i) = 1 + \dim(a)$ *) for all* $i \in \{1, 2, 3\}$.

Proof. If $n = 2$ or $\dim(a) = -1$ or $\dim(c) = n$ this reduces to the previous axiom. So suppose $n > 2$ and $\dim a = k > -1$, see figure 2.9. By axiom 2.8.1 there is a point $p \prec c$, $p \not\preceq a$. By axiom 2.8.2 $b_1 = a \vee p$ has dimension $k + 1$. Again there is a point $q \prec c$, $q \not\preceq b_1$. Define $l = p \vee q$. This has at least a third point r. Suppose $r \preceq a$. Then $r \prec b_1$, hence $l \prec b_1$, hence $q \prec b_1$ †. Define $b_2 = a \vee q$, $b_3 = a \vee r$. Equality of two of the b_i again leads to a contradiction. ⋄

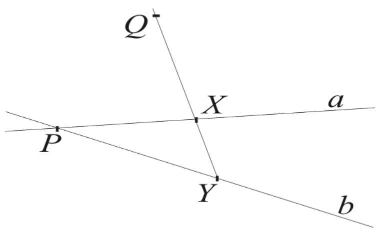

Figure 2.10: Perspective lines

Proposition 2.9.3 *All lines contain an equal number of points.*

Proof. First suppose the lines a, b have a common point $a \wedge b = P$, see figure 2.10. Then by the dimension theorem $a \vee b = \alpha$ is a plane. We will first show that there is at least one point Q in α outside a and b. Take a point $C \neq P$ on a and $D \neq P$ on b. $l = C \vee D$ is a line which must have a third point Q on it. If Q is on a (or on b) then $a = l$ (or $b = l$). †
Next, for X on a and Y on b, the maps

$$f : X \to (X \vee Q) \wedge b$$

and

$$g : Y \to (Y \vee Q) \wedge a$$

are bijections from a to b resp. from b to a and inverse to each other.
If $a \wedge b = \mathbf{0}$, i.e. they are skew, take a point P on a and Q on b and let $c = PQ = P \vee Q$, which is a line. Then we can construct bijections between a and c and between b and c. $\diamond$

Corollary 2.9.4 *Let $a \prec c$ and $\dim(c) = 2 + \dim(a)$. Then the number of elements b between a and c is the same as the number of points on a line (all 1-dimensional open intervals have the same number of elements).*$\diamond$

Corollary 2.9.5 *Let be given the elements x and y, both not equal to $\mathbf{1}$. Then there exists a point that is contained in neither. And, dually, if $x \neq \mathbf{0} \neq y$ then there is a hyperplane that contains neither.*

Proof. We distinguish the following cases. (1) $x \preceq y$. Since $y \prec \mathbf{1}$, from axiom 2.8.1 follows that there is a point p that is not contained in y, hence not in x. (2) $y \prec x$ similar. (3) Otherwise. Now $x \wedge y \prec x$ and $x \wedge y \prec y$. Since $x \prec \mathbf{1}$ there is a point p not in x. If $p \not\preceq y$ then p satisfies our proposition. If $p \preceq y$ then from axiom 2.8.1 again we know that there is a point $q \preceq x$ with $q \not\preceq x \wedge y$. The line $p \vee q$ contains a third point r. If $r \preceq x$ then this whole line is in x, including p, †. Likewise r cannot be in y. Hence r satisfies our proposition. $\diamond$

Corollary 2.9.6 *If neither $x \preceq y$ nor $y \preceq x$ then there is a point $p \prec x \vee y$ that is neither in x nor in y.*

Proof. This follows from the last part of the previous proof. $\diamond$

At last we can give the definition of a projective space.

Definition 2.9.7 *A* projective space *is a quadruple* $(S, n, \dim, I)$ *in which* S *is a set,* $n \geq 3$ *is an integer,* $\dim : S \to \{-1, 0 \ldots n\}$ *is a surjective function, and* I *is a binary relation on* S*, satisfying*

- *the axiom of order, 2.2.1:*
  - $(x, x) \in I$
  - $(\ (x, y) \in I \quad and \quad (y, x) \in I\ ) \ \Rightarrow\ x = y$
  - $(\ (x, y) \in I \quad and \quad (y, z) \in I\ \ ) \ \Rightarrow\ (x, z) \in I$
- *the axiom of monotone dimension, 2.2.7: for every* $x, y \in S$ *we have:*
$$x \prec y \Rightarrow \dim(x) < \dim(y)$$
- *2.3.1 the axiom of border:*
  - *there is an element* $\mathbf{0}$ *such that for every* $x \in S : \mathbf{0} \preceq x$*;*
  - *there is an element* $\mathbf{1}$ *such that for every* $x \in S : \mathbf{1} \succeq x$
- *2.4.3, the Lattice axiom:*
  - *each pair of elements of* S *has a least upper bound*
  - *each pair of elements of* S *has a greatest lower bound*
- *2.8.1 the axiom of sufficient points/hyperplanes:*
  - *for every pair* a, b *of elements of* S *for which* $a \prec b$*, there exists a point* x *such that* $x \not\preceq a$ *and* $x \preceq b$
  - *for every pair* a, b *of elements of* S *for which* $a \prec b$*, there exists a hyperplane* y *such that* $a \preceq y$ *and* $b \not\preceq y$
- *2.8.2 the axiom of composition*
  - *if* x *is a point and* a *any element not containing* x*, then* $\dim(a \vee x) = \dim(a) + 1$
  - *if* y *is a hyperplane and* b *any element not in* y*, then* $\dim(b \wedge y) = \dim(b) - 1$
- *2.9.1 the axiom of cardinality:*
  - *every line has at least three points on it*
  - *every dual line is contained in at least three hyperplanes*

In addition, 1- and 2-dimensional intervals of higher dimensional projective spaces, are called projective spaces as well[3]*.*

---

[3] We deliberately exclude 0-dimensional spaces. Such a space would be for instance $[P, l]$, where P is a point on line l. It contains two elements. An interval $[x, x]$, with only one element would get dimension -1. Anyhow, they have no practical geometrical value.

Note 1. If one wants to develop ordinary real projective geometry, one more axiom is required: explicit statement that the ground field is **R** (see section 4.9). That implies the axiom of cardinality, however.

Note 2. Our axioms are certainly not independent. The desire to formulate them symmetrically to ensure duality, prevents this. That they are not contradictory follows from example 1 on page 28. We leave it to the logicians to find minimal subsets of independent axioms.

As a first and very important result we formulate the theorem of

Proposition 2.9.8 Desargues
Let be given two triangles consisting of six different points A, B, C, A', B', C' *and six different lines* $a = BC,\ b = CA,\ c = AB,\ a' = B'C',\ b' = C'A',\ c' = A'B'$. *Then*

$$\dim(AA' \wedge BB' \wedge CC') = 0$$

if and only if

$$\dim((a \wedge a') \vee (b \wedge b') \vee (c \wedge c')) = 1$$

Note that if either condition is met, the whole configuration is contained in a 2- or 3-dimensional blade, see figure 2.11. Hence, for a proof we can refer to any elementary book on projective geometry.

We say that the triangles are *perspective* from a point (viz. O) if and only if they are perspective from a line (viz. l).

Exercise. Formulate the dual of the Desargues statement. ⋄

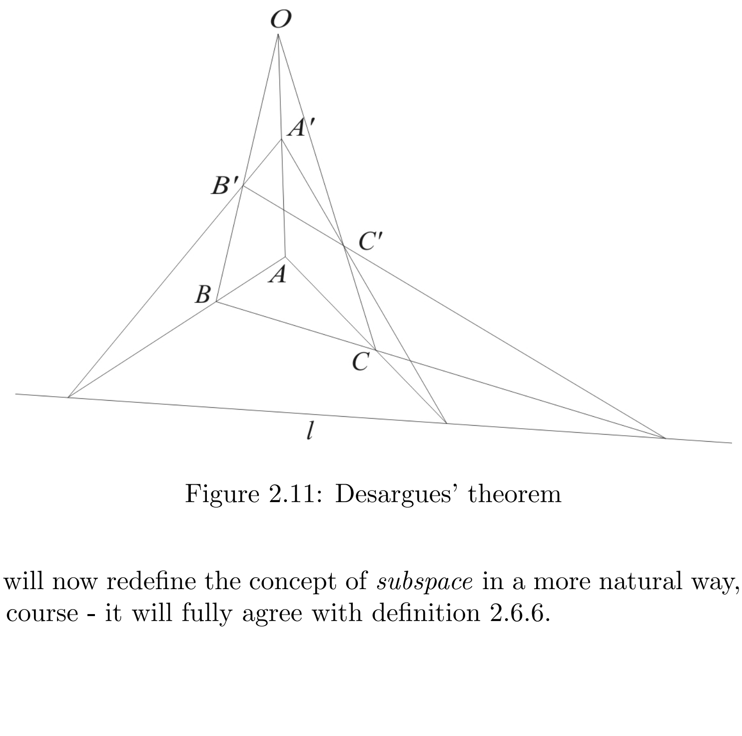

Figure 2.11: Desargues' theorem

We will now redefine the concept of *subspace* in a more natural way, but - of course - it will fully agree with definition 2.6.6.

Definition 2.9.9 *Let* $(S, n, \dim, I)$ *and* $(S', n', \dim', I')$ *be projective spaces (of dimension 1 or more). The second is called a* subspace *of the first if*

- $S' \subseteq S$
- *for every* $x, y \in S' :\ (x, y) \in I \ \Leftrightarrow\ (x, y) \in I'$
- *for every* $x, z \in S'$ *and for every* $y \in S :\ x \prec y \prec z \ \Rightarrow\ y \in S'$ *(completeness of blades)*

Note that I' is the restriction of I to $S' \times S'$.

Proposition 2.9.10 *Let* $(S', n', \dim', I')$ *be a subspace of* $(S, n, \dim, I)$*. Then there exists an integer* $a \geq 0$ *with the property that for every* $x \in S'$ *we have* $\dim'(x) = \dim(x) - a$*.*

Proof. Let $0'$ be the lower border of S'. Then there is an integer $a \geq 0$ and a composition chain

$$0 = p_0 \prec \cdots \prec p_k \prec \cdots \prec p_a = 0'$$

Also there is a composition chain

$$0' = p_a \prec' \cdots \prec' p_{a+k} \prec' \cdots \prec' p_{a+m} = x$$

where $m = \dim'(x) + 1$. But the last chain is also a composition chain in S, so we have the composition chain

$$0 = p_0 \prec \cdots \prec p_k \prec \cdots \prec p_{a+m} = x$$

and hence $\dim(x) = a + m - 1 = a + \dim'(x)$. $\diamond$

Corollary 2.9.11 *It follows that* $n' \leq n$*. In addition, the new definition of subspace is, indeed, in accordance with definition 2.6.6.*$\diamond$

Definition 2.9.12 S' *is a* proper *subspace of* S *if it is a subspace and if it is in addition a proper subset of* S *or - equivalently - if* $n' < n$*.*

Proposition 2.9.13 *Let* $(S, n, \dim, I)$ *and* $(S', n', \dim', I')$ *be projective spaces. The following propositions are equivalent:*

- S' *is a subspace of* S
- *there are three elements* $x \prec y \prec z$ *of* S *such that* $S' = [x, z]$
- S' *is an interval of* S*, of dimension 1 or more.* $\diamond$

We close this section with two examples.

Example 1. *The lattice of subspaces of a vector space*. Let V be the $(n+1)$-dimensional left vector space over a skew field F, with $n \geq 3$. Let S_k be the set of k-dimensional linear subspaces of V and $S = \bigcup_{k=0}^{n} S_k$. Define $\dim(L) = k - 1$ for every $L \in S_k$, then dim is a surjective map from S to $\{-1, 0, \ldots n\}$. Clearly, 1-dimensional subspaces are called ‘points’ now, and 2-dimensional ones ‘lines’. Define $L \preceq M$ by $L \subseteq M$, then this relation satisfies axioms 2.2.1 and 2.2.7. The borders are $\{0\}$ and V, satisfying 2.3.1. The meet of two subspaces is simply their intersection, and the join their direct sum, satisfying 2.4.3. Also the two remaining axioms hold. The dimension theorem was already a property of subspaces, but is shown once more. The last axiom, 2.9.1, needs some reflection. If the characteristic k of the field is 0, there are infinite points on each line. Otherwise the number of vectors on a 1-dimensional subspace equals k, so the number of ‘points’ on a ‘line’ is $k + 1 > 2$.

Nowadays, projective space is usually defined as the quotient of a vector space V by the relation $v \sim \lambda v$, where $v \in V$ and λ any non-zero scalar. But that gives only the points of the projective space. So we have to extend this definition in the above way, in order to deal with lines etc. too. $\diamond$

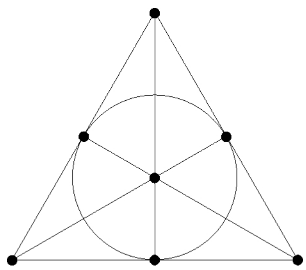

Figure 2.12: The Fano plane

Example 2. *The Fano space*. The Fano 3-space is the collection of linear subspaces of the vector space $\mathbf{F}_2^4$, the 4-dimensional space over the finite field $\mathbf{F}_2$. It consists of 15 points, 35 lines and 15 planes. Each plane - in turn called a Fano plane - contains 7 points and 7 lines, see figure 2.12. Verify that this plane satisfies the definition of a projective space and try to compose an image of the space. Concerning the Desargues statement: no two different triangles satisfy the conditions of the proposition, hence the statement is trivially true. In fact this means that it is not necessary to require that the Fano plane can be embedded in a 3-dimensional space. $\diamond$

2.10 3-dimensional geometry

We are now in a position to prove the 'normal' axioms and basic theorems of 3-dimensional projective geometry. Doing this is more than just an example. It will enable us to loosen our language and thus to use 'everyday' expressions and facts in the sequel. So, in this section S is a 3-dimensional projective space.

We refer to the axioms of incidence in section 2.1 of [Coxeter]. Coxeter's axiom 2.111 and 2.113 follow directly from axiom 2.9.1. His 2.114 and 2.116 are specializations of our 2.8.1.

Proposition 2.10.1 *Two distinct points determine one line that contains them.*

This is [Coxeter] 2.112. A precise formulation would be: *Let V_k be the set of k-blades, then*

$$\forall\ P \in V_0\ \forall\ Q \in V_0 \setminus \{P\}\ \exists!\ l \in V_1 :\ P \prec l\ \&\ Q \prec l$$

The dual statement is [Coxeter] 2.117.

Proof. Let $P \neq Q$ be the points and $l = P \vee Q$. Because the points are different we have $P \wedge Q = \emptyset$, hence $\dim l = \dim(P \vee Q) = \dim P + \dim Q - \dim \emptyset = 0 + 0 + 1 = 1$. So l is a line. If m is a line that contains P and Q, it contains l, hence $l = m$. $\diamond$

Proposition 2.10.2 *A point and a line not through that point determine a unique plane.* $\diamond$

Proposition 2.10.3 *Two different lines that have one common point determine a unique plane.* $\diamond$

Proposition 2.10.4 *Two different lines that are in one plane have one point in common.* $\diamond$

Proposition 2.10.5 *A plane and a line not in that plane have one point in common.*

Proof. Let α be the plane, l the line and $P = l \wedge \alpha$. We will show that $\dim P = 0$. From its definition we know already that $\dim P \leq 1$. If it were -1 then $\dim l \vee \alpha = 1 + 2 + 1 = 4$, which is impossible in 3-space. If it were 1 then $P = l \prec \alpha$, contradicting the hypothesis. $\diamond$

Proposition 2.10.6 *Let A, B and C be different non-collinear points. Let $D \prec BC$ but $B \neq D \neq C$ and $E \prec AC$ but $A \neq E \neq C$. Then there is a point $F \prec AB$ such that D, E, F are collinear.*

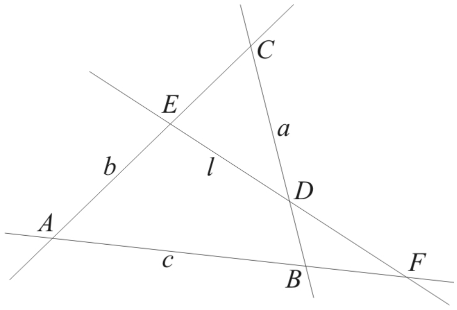

Figure 2.13: Proposition 2.10.6

This is [Coxeter] 2.115.

Proof. Let $a = BC = B \vee C$, $b = CA$, $c = AB$, $l = DE$, see figure 2.13. Then $\alpha = A \vee a = A \vee B \vee C$ is a plane. Since D and E are also in α, $l \prec \alpha$. If $l = c$ then $C \prec c$, †. But then l and c are different lines in a plane, hence have a common point F. Then F satisfies the conditions of our proposition. $\diamond$

Chapter 3

Isomorphisms

The first section of this Chapter we owe again to Jacobson. The remaining part gives a reformulation of the basic concepts of perspectivity and homology.

3.1 Maps

In this section we deal with more than one space, and the structures of them are compared. Let S and S' be projective spaces, each of dimension 2 or more[1].

Definition 3.1.1 *A map* $f : S \to S'$ *is called* order preserving *or* monotone *if for every* $a, b \in S$*:*

$$a \preceq b \quad \Rightarrow \quad f(a) \preceq' f(b)$$

Examples. 1. The identity map $1_S : S \to S$ is, of course, order preserving, and so is the constant map $f_a : S \to S'$, where a is any element of S' and $f(x) = a$ for every $x \in S$.

2. If $[a, b]$ is a subspace of S then the *embedding* $f : [a, b] \to S$, defined by $f(x) = x$, is order preserving too.

3. Now let P be a point on a line l in the ordinary projective plane α. Let the *projection* $f : [\emptyset, \alpha] \to [\emptyset, l]$ be defined as follows:

- for $x \in [\emptyset, l] : \; f(x) = x$

[1] It is not necessary to require dimension 3 or more since in the 2-dimensional case we know that these spaces can be embedded in 3-dimensional ones.

- for each point $X \prec \alpha$ that is not on l : $f(X) = P$.
- for each line $m \prec \alpha$: $f(m) = l$
- $f(\alpha) = l$

It is easy to verify that this map is oder preserving. $\diamond$

Definition 3.1.2 *A map* $f : S \to S'$ *is called a* homomorphism *if for every* $a, b \in S$*:*

$$f(a \vee b) = f(a) \vee' f(b)$$

and

$$f(a \wedge b) = f(a) \wedge' f(b)$$

If no confusion is possible we will omit the primes of the symbols $\preceq, \wedge$ and $\vee$.

Proposition 3.1.3 *Each homomorphism is order preserving.*

Proof. $a \preceq b \Rightarrow a \wedge b = a \Rightarrow f(a) \wedge f(b) = f(a \wedge b) = f(a) \Rightarrow f(a) \preceq f(b)$. $\diamond$

The projection map in example 3 above is not a homomorphism. For take any two different points A and B in α that are not both on l. Then $m = A \vee B$ is a line. Now $f(A \vee B) = f(m) = l$ whereas $f(A) \vee f(B) = P \vee P = P$. So the converse of proposition 3.1.3 is not true.

Definition 3.1.4 *An* isomorphism *or* projectivity *or* projective map *is a bijective homomorphism. The spaces* S *and* S' *are called* isomorphic *if there exists an isomorphism between them.*

Proposition 3.1.5 *A map* $f : S \to S'$ *is an isomorphism if and only if it is bijective and for every* $a, b \in S$*:*

$$a \preceq b \iff f(a) \preceq f(b)$$

Proof. If f is an isomorphism then by definition f is a homomorphism and hence order preserving. If $f(a) \preceq f(b)$ then $f(a \wedge b) = f(a) \wedge f(b) = f(a)$ so $a \wedge b = a$ and hence $a \preceq b$. This proves half of the statement.
Conversely let f be bijective and for every $a, b \in S : a \preceq b \iff f(a) \preceq f(b)$. Since $a \wedge b \preceq a$ and $a \wedge b \preceq b$, also $f(a \wedge b) \preceq f(a)$ and $f(a \wedge b) \preceq f(b)$, hence $f(a \wedge b) \preceq f(a) \wedge f(b)$. Now let $x \in S'$ be a lower bound of $f(a)$ and $f(b)$, so $x \preceq f(a) \wedge f(b)$, and let $y = f^{inv}(x)$. Clearly $y \preceq a$, $y \preceq b$ and $y \preceq a \wedge b$. Then again $x \preceq f(a \wedge b)$ which means that $f(a \wedge b)$ is greatest lower bound of $f(a)$ and $f(b)$, hence equal $f(a) \wedge f(b)$. Analogous for $\vee$. $\diamond$

Of course the inverse of an isomorphism is again an isomorphism.

Proposition 3.1.6 *Isomorphic spaces have equal dimensions.*

Proof. Let $f : S \to S'$ be an isomorphism. Note that $a \prec b \Leftrightarrow f(a) \prec f(b)$. Let $\mathbf{0} \prec a_0 \prec \ldots \prec a_{n-1} \prec \mathbf{1}$ be a composition chain. Then $f(\mathbf{0}) \prec f(a_0) \prec \ldots \prec f(a_{n-1}) \prec f(\mathbf{1})$. That means that the dimension of S' is at least equal to that of S. Analogously $\dim S \geq \dim S'$. $\diamond$

Let G denote the set of *automorphisms* of a projective space S, that is the set of isomorphisms of S to itself. Then $(G, \circ)$ is a group.

Important 'quasi isomorphisms' are the following (remember that the dimension of S is at least 2).

Definition 3.1.7 *A* correlation *or* contravariant projectivity *is a bijective map $f : S \to S$ that reverses order, i.e. for every $a, b \in S$:*

$$a \preceq b \;\Leftrightarrow\; f(a) \succeq f(b)$$

By contrast we have:

Definition 3.1.8 *A* collineation *or* covariant projectivity *is a bijective map $f : S \to S$ that preserves order.*

Hence a collineation is just an automorphism.

Correlations can in fact be extended to isomorphisms by composing them with the 'identity' between S and its dual. To be precise, let $1^*_S : S \to S^*$ be defined by $1^*_S(x) = x$. Obviously 1^*_S is order reversing and $1^*_S \circ f$ is an isomorphism for each correlation f.

Proposition 3.1.9 *If f is a correlation then for every $a, b \in S$:*

$$f(a \vee b) = f(a) \wedge f(b)$$

and

$$f(a \wedge b) = f(a) \vee f(b). \diamond$$

The collineations of S form the above group, G. The correlations of S do *not* form a group: two correlations obviously compose to a collineation, or even (without proof), every collineation can be factored into two correlations. Thus if H is the group of all correlations and collineations, we have that G is a subgroup of H of index 2.

So far we have avoided 1-dimensional spaces. These require special care, as the following example will show. Let S be the real projective line, consisting of one line, its (infinite) points and the empty set. Take any two different points a, b and let $f(a) = b$, $f(b) = a$ and $f(x) = x$ in all other cases, including $\mathbf{0}$ and $\mathbf{1}$. This map surely preserves order, join and meet. But it is not a projective map according to the common definition.

Definition 3.1.10 *Two 1-dimensional spaces S, S' are* isomorphic *if they are contained in 2-dimensional spaces U, U' respectively, and if there exists an isomorphism $f : U \to U'$ with $f(S) = S'$. A map $g : S \to S'$ is an* isomorphism *if it can be extended to an isomorphism $f : U \to U'$.*

3.2 Perspectivities

Let S be an n-dimensional projective space, $n \geq 2$, $[a, b]$ a k-dimensional subspace, and $c \in S$ arbitrary. Hence $a \prec b$ and $1 \leq k \leq n$. Consider the map $f : [a, b] \to S$ defined by $f(x) = x \vee c$. The image of $[a, b]$ does not exceed $[a \vee c, b \vee c]$, so we redefine

$$f : [a, b] \to [a \vee c, b \vee c]$$

This map is obviously order preserving. It is surjective too: for either $a \vee c = b \vee c$, in which case we deal with a constant, surjective map; or else there is for every $y \in \langle a \vee c, b \vee c]$ an integer $j > 0$ and points $p_1, \dots p_j$ such that

$$a \vee c \prec a \vee c \vee p_1 \prec \cdots \prec a \vee c \vee p_1 \vee \cdots \vee p_j = y$$

is a composition chain. But then $f(a \vee p_1 \vee \cdots \vee p_j) = y$.

Under what conditions is f injective? If it is injective, an obvious candidate for an inverse of f is the map

$$g : [a \vee c, b \vee c] \to [a, b]$$

defined by $g(y) = y \wedge b$. This seems to violate duality. But if we put $p = a \vee c$ and $q = b \vee c$ we have $f(x) = x \vee p$ and $g(y) = y \wedge b$, see figure 3.1.

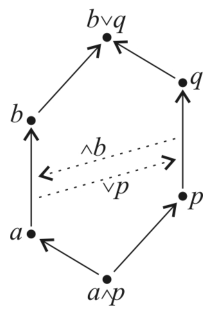

Figure 3.1: Dual perspectivities

Composing the two gives

$$g \circ f(x) = (x \vee c) \wedge b$$

$$f \circ g(y) = (y \wedge b) \vee c$$

But there is no guarantee that these equal the identity maps.

Let's first consider two extreme cases.

- if $c = \mathbf{0}$ then $f(x) = x \vee \mathbf{0} = x$, hence $f = 1_{[a,b]} = g$ is bijective for all admitted pairs a, b.
- if $c = \mathbf{1}$ then $f(x) = x \vee \mathbf{1} = \mathbf{1}$, hence f is constant and not injective, for all admitted a, b.

But it is not so much the position of c as well as that of $b \wedge c$ relative to a and b that determines whether or not f is injective. We have three cases (obviously $b \wedge c \preceq b$):

1 $b \preceq (b \wedge c)$

2 $a \prec (b \wedge c) \prec b$

3 $(b \wedge c) \preceq a$

The first case, $b \preceq (b \wedge c)$, implies $b \preceq c$. We have $f(x) = c$, a constant map again, and not injective; $c = \mathbf{1}$ is a special case.

If, in the second case, $a \prec (b \wedge c) \prec b$ then $a \preceq c$ hence $f(a) = a \vee c = c$. But also $f(b \wedge c) = c$, so f is not injective.

That leaves the third case. So suppose $(b \wedge c) \preceq a$. From proposition 2.7.1 we know that

$$\begin{aligned} g \circ f(x) \quad &= (x \vee c) \wedge b \succeq (x \wedge b) \vee (c \wedge b) = \\ &= x \vee (c \wedge b) = x \end{aligned}$$

In addition we have
1. $a \preceq x$ hence $a \wedge c \preceq x \wedge c \preceq c$
2. $x \preceq b$ hence $x \wedge c \preceq b \wedge c \preceq a$
3. from 1 and 2: $x \wedge c \preceq a \wedge c$
4. from 1 and 3: $a \wedge c = x \wedge c$.
By the dimension theorem we have

$$\begin{aligned} \dim(f(x)) \quad &= \dim(x) + \dim(c) - \dim(x \wedge c) \\ &= \dim(x) + \dim(c) - \dim(a \wedge c) \end{aligned}$$

Next consider $g(y) = y \wedge b$ with $y \in [a \vee c, b \vee c]$. Now we have $y \preceq b \vee c$ and $b \preceq b \vee c$ hence $y \vee b \preceq b \vee c$. Also $c \preceq y \preceq y \vee b$ and $b \preceq y \vee b$ hence $b \vee c \preceq y \vee b$. So also $y \vee b = b \vee c$. Then

$$\begin{aligned}\dim(g(y)) &= \dim(y) + \dim(b) - \dim(y \vee b)\\ &= \dim(y) + \dim(b) - \dim(b \vee c)\end{aligned}$$

If we now substitute $y = f(x)$ we get

$$\begin{aligned}\dim(g(f(x))) &= \dim(f(x)) + \dim(b) - \dim(b \vee c)\\ &= \dim(x) + \dim(c) - \dim(a \wedge c) + \dim(b) - \dim(b \vee c)\\ &= \dim(x) + \dim(b \wedge c) - \dim(a \wedge c)\\ &= \dim(x)\end{aligned}$$

The last equality holds because - by hypothesis - $b \wedge c \preceq a$. Now we have both $g \circ f(x) \succeq x$ and $\dim(g(f(x))) = \dim(x)$, so $g \circ f(x) = x$, indeed. Since f is surjective and has a left inverse, it is injective as well. Does this mean that f is an isomorphism? Yes, as long as the dimension of $[a, b]$ is at least 2. For 1-dimensional intervals we need special care.

Let $[a, b]$ have dimension 1. We know that this interval is in a space S of dimension 3 or more. Let $c \in S$ have the required property $b \wedge c \preceq a$, and let $f(x) = x \vee c$. If $b = \mathbf{1}$ then $c \preceq a$ hence f is the identity on $[a, b]$, hence an isomorphism. Next suppose $b \neq \mathbf{1}$. Then c cannot be $\mathbf{1}$ either. In that case there is a point p with $p \npreceq b$ and $p \npreceq c$. Then $[a, b \vee p]$ has dimension 2 and $(b \vee p) \wedge c \preceq a$. So the map f can be extended to an isomorphism from $[a, b \vee p]$ to $[a \vee c, b \vee p \vee c]$.

At last we have proved:

Proposition 3.2.1 *The map f is an isomorphism if and only if $b \wedge c \preceq a$.*

Dually we have

Proposition 3.2.2 *The map $f : [a, b] \to [a \wedge c, b \wedge c]$ defined by $f(x) = x \wedge c$ is an isomorphism if and only if $a \vee c \succeq b$.* $\diamond$

Exercise. Prove this proposition directly. $\diamond$

Definition 3.2.3 *In this book*[2] *non trivial isomorphisms $x \mapsto x \wedge c$ and $x \mapsto x \vee c$ are called* perspectivities.

Exercise. Remember that in real projective 3-space there are 6 types of subspaces: $[\emptyset, l]$, $[\emptyset, \alpha]$, $[\emptyset, \mathbf{P}_3]$, $[A, \alpha]$, $[A, \mathbf{P}_3]$, and $[l, \mathbf{P}_3]$, where we presume $A \prec l \prec \alpha$. List the 10 types of perspectivities. $\diamond$

[2] Normally the word *perspectivity* is used for bijective maps $x \mapsto (x \vee a) \wedge b$ and $x \mapsto (x \wedge a) \vee b$

We will show that all k-dimensional subspaces of one space are isomorphic. But first we restrict to k-blades.

Proposition 3.2.4 *For each pair of k-blades a and b in S, there exists an isomorphism $f : [\mathbf{0}, a] \to [\mathbf{0}, b]$.*

Proof. [1] If $a = b$ we take the identity. If not, $c = a \wedge b$ has dimension $l < k$. Let $d = a \vee b$, which has dimension $2k - l$. [2] By proposition 2.9.5 there is a point $r_1 \prec d$, neither in a nor in b. Define $a_1 = a \vee r_1$ and $b_1 = b \vee r_1$, both having dimension $k + 1$. Note that $a_1 \vee b_1 = d$ and $\dim(a_1 \wedge b_1) = l + 2$. If $a_1 = d$ (and hence $b_1 = d$) we go on with [3], below. If not, there is a point $r_2 \prec d$, $r_2 \not\preceq a_1$, $r_2 \not\preceq b_1$. Define $a_2 = a_1 \vee r_2$, $b_2 = b_1 \vee r_2$, which have dimension $k + 2$. Go on until - after a finite number of steps - we arrive at $d = a_j = a_{j-1} \vee r_j = a \vee r_1 \vee \cdots \vee r_j = b_j$. Evidently $2k - l = k + j$, so $j = k - l$. [3] Define $r = r_1 \vee \cdots \vee r_j$, which has dimension $j - 1 = k - l - 1$. Then by the dimension theorem $r \wedge a = r \wedge b = \mathbf{0}$. Define the following perspectivities (check!):

$$f_1 : [\mathbf{0}, a] \to [r, d] \text{ by } f_1(x) = x \vee r$$
$$f_2 : [r, d] \to [\mathbf{0}, b] \text{ by } f_2(x) = x \wedge b$$

Then $f = f_2 \circ f_1$ is an isomorphism. $\diamond$

Dually we have:

Proposition 3.2.5 *For each pair of k-blades a and b in S, there exists an isomorphism $f : [a, \mathbf{1}] \to [b, \mathbf{1}]$.* $\diamond$

Exercise. Prove this statement directly. $\diamond$

At last we arrive at the general result.

Proposition 3.2.6 *Two subspaces S_1 and S_2 of S are isomorphic if and only if they have equal dimensions.*

Proof. Remember that 3.1.6 is half of this proposition, so we only have to prove that subspaces of equal dimensions are isomorphic. Let $[a, b]$ be a k-dimensional subspace of S. There is only one subspace, viz. S, of dimension n, so for $k = n$ there is nothing to prove. Suppose $1 \leq k < n$. We will prove that $[a, b]$ is isomorphic to a k-dimensional subspace of the form $[c, \mathbf{1}]$ or $[\mathbf{0}, c]$. Since $k < n$ either $a \succ \mathbf{0}$ or $b \prec \mathbf{1}$ or both. Suppose first $b \prec \mathbf{1}$. Then there is a point $p_1 \not\preceq b$ and a perspectivity $x \mapsto x \vee p_1$ that lifts our interval. If $b \vee p_1 = \mathbf{1}$ we are done. If not we can repeat the process with additional points until at last, after a finite number of steps we arrive at $b \vee p_1 \vee \cdots \vee p_j = \mathbf{1}$. Then $[a, b]$ is isomorphic with $[a \vee p_1 \vee \cdots \vee p_j, \mathbf{1}]$. The case $b = \mathbf{1}$, $a \succ 0$ is left as an exercise. $\diamond$

3.3 Homologies

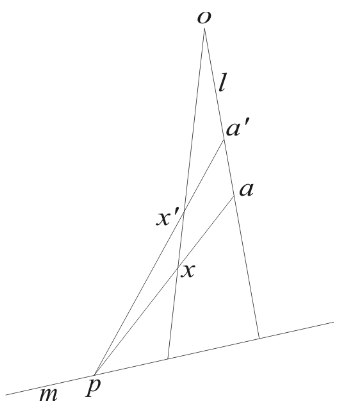

Figure 3.2: Homology

Let α be any hyperplane of a n-dimensional projective space S. Take three different points o, a and a' on one line l and not in α, so $l \not\preceq \alpha$ (note that we need at least 4 points on l; if there are only three, we can do the next construction with $a = a'$, which will give the identity). We are going to define an elementary projective map $f : S \to S$, called a *homology*. First it will be defined for points, and afterwards extended to all other elements of S. So, take an arbitrary point $x \not\preceq l$. Define $f(x) = (((x \vee a) \wedge \alpha) \vee a') \wedge (o \vee x)$, see figure 3.2, where $p = (a \vee x) \wedge \alpha$ and $x' = f(x)$. Note that the lines l, $o \vee x$, $p \vee a$ and $p \vee a'$ are all in one plane β, that β meets the hyperplane α in a line m and that $x' \neq o$. In addition, if $x \prec \alpha$ then $f(x) = x$ and if not $f(x) \neq x$.

For a point $x \prec l$ we use an intermediate point y, $y \not\preceq l$, $y \not\preceq \alpha$, see figure 3.3. Construct $y' = f(y)$ as above. Since $y \not\preceq \alpha$, $y' \neq y$. Define $x' = f(x) = (((x \vee y) \wedge \alpha) \vee y') \wedge l$, which, again, is a point. Note that $f(o) = o$, $f(a) = a'$ and $f(r) = r$, where $r = l \wedge \alpha$. Note also that it is not necessary that the line has at least 6 points, as in the figure. It could have three points, then $a = a'$ and f is the identity, or four or five in which cases several points coincide. (Five points is in fact impossible, but for the moment we cannot yet exclude this.)

Proposition 3.3.1 *This definition is independent of the choice of y.*

Proof. Take another such point, z, and construct $z' = f(z)$, see figure 3.4. We have to prove that $x' \prec (s \vee z')$, or, equivalently, $s \prec (x' \vee z')$. Let $\beta = y \vee l$ and $\gamma = z \vee l$. In the figure we have $\beta \neq \gamma$, but the proof of the other case is exactly the same. The triangles ayz and $a'y'z'$

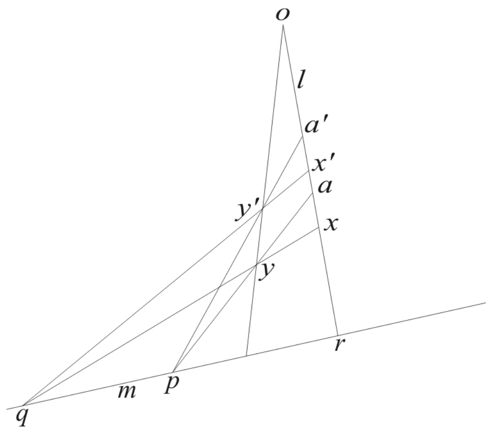

Figure 3.3: Homology on l

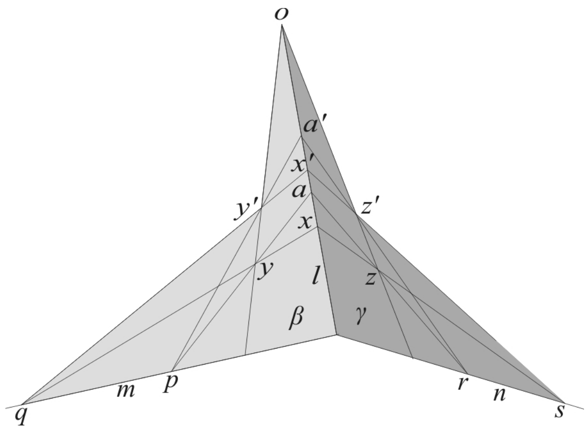

Figure 3.4: Consistency of homology

are perspective from o, hence from $p \vee r$ (Desargues, 2.9.8). So, $u = (y \vee z) \wedge (y' \vee z')$ is a point on $p \vee r$. Next the triangles xyz and $x'y'z'$ are perspective from point o, hence from $q \vee u$. That means that $s \prec (x' \vee z')$. $\diamond$

Verify that if g is the map defined as above, but with a and a' interchanged, then $g \circ f = f \circ g = 1$ on the set of points, hence f is bijective on this set.

Proposition 3.3.2 *Collinear points are mapped on collinear points.*

Proof. Since α is pointwise invariant, all its subspaces, including its

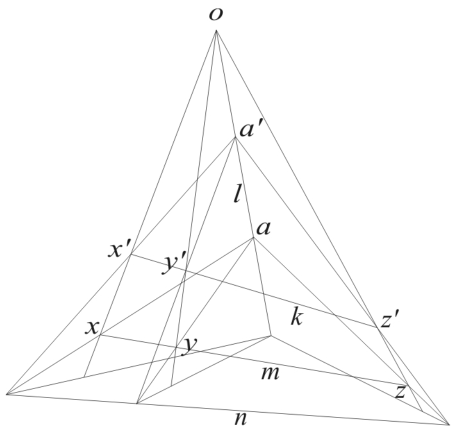

Figure 3.5: Collinear points

lines, are invariant. That is, the proposition holds in α. So, let $x,\ y,\ z$ be on line $m \not\preceq \alpha$. Suppose first that $m \wedge l = \mathbf{0}$. Then $a \vee m$ is a plane that meets α in a line n. Again, this configuration is in a 3-dimensional subspace. The planes $n \vee a'$ and $m \vee o$ meet in a line k. By construction, $x',\ y',\ z'$ are on k. All other possibilities are left as an exercise for the reader. $\diamond$

So, the f-image of a line is defined as the join of the images of any pair of its points. Of course we define $f(\mathbf{0}) = \mathbf{0}$. Now f preserves incidence in each line: for each line l and each point $p \prec l$ we have $f(\mathbf{0}) \prec f(p) \prec f(l)$. Trivially f preserves meet and join.

Suppose we have extended the definition of f to all elements of dimension $< k$, and suppose for all these elements x f preserves meet and join on $[\mathbf{0}, x]$. We will define f on k-blades.

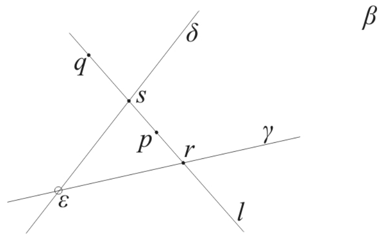

Figure 3.6: The image of k-blades

Let β be a k-blade, $\gamma \prec \beta$ a $(k-1)$-blade and $p \prec \beta$ a point not in γ. Note that $\beta = \gamma \vee p$. We define $f(\beta) = f(\gamma) \vee f(p)$. This is a k-blade again.

We have to show that this definition does not depend on the choice of γ and p. So, suppose $\delta \vee q = \beta$ with q a point and $\dim(\delta) = k - 1$, see figure 3.6. Define $\epsilon = \gamma \wedge \delta$, $l = p \vee q$, $r = l \wedge \gamma$ and $s = l \wedge \delta$. Suppose first $\gamma \neq \delta$ and $p \neq q$. Then ϵ is a $(k-2)$-blade, l is a line and r and s are points. We include the possibilities
- $r = s$
- $p = s$
- $q = r$
- $p = s$ and $q = r$.

If we write x' for $f(x)$ etc. throughout, we have $\beta' \overset{1}{=} p' \vee \gamma' \overset{2}{=} p' \vee r' \vee \epsilon' \overset{2}{=} l' \vee \epsilon' \overset{2}{=} q' \vee s' \vee \epsilon' \overset{2}{=} q' \vee \delta'$.
1 is by definition and 2 by hypothesis.
The reader is invited to check the other configurations ($p = q$ or $\gamma = \delta$).
If $\beta = \mathbf{1}$ there is - of course - but one possibility: $f(\mathbf{1}) = \mathbf{1}$.

Thus, we extended f to S. In the same way we can extend its 'inverse' g and obviously we have $gf = fg = 1_S$. That is, they are isomorphisms.

Definition 3.3.3 *A* homology *is an automorphism that leaves a hyperplane and a point not in that hyperplane invariant.*

We summarize the results in:

Proposition 3.3.4 *A homology f is completely determined by its invariant point o and hyperplane α, and one additional pair $x, f(x)$, provided that $o \not\leq x \not\leq \alpha$, $o \not\leq f(x) \not\leq \alpha$, $\dim x = \dim f(x)$, $o \vee x = o \vee f(x)$ and $\alpha \wedge x = \alpha \wedge f(x)$. If there are only three points on each line, there is only one homology, viz. the identity. If there are k points on each line then there are $k - 2$ homologies for each pair of fixed elements.* $\diamond$

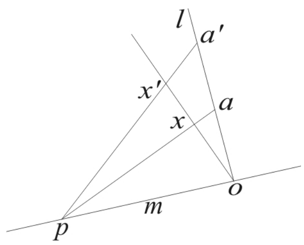

Figure 3.7: Elation

In the beginning of this section we required that o is not in α. If o is in α we can do a similar construction, see figure 3.7. In that case f is called an *elation*. We leave the details as an exercise for the reader.

The homologies do not form a subgroup of the automorphisms, nor do the elations, as is easy to check. However, the homologies (elations) that keep o and α fixed do.

Let us look at the group G of homologies that fix o and α. The map that moves a to a' is denoted by $f_{a,a'}$. This makes sense only if a, a', o are collinear. For each $a \not\leq \alpha$, $a \neq o$ we have $f_{a,a} = 1_S$. What does it mean when $f_{a,a'} = f_{b,b'}$? We will show

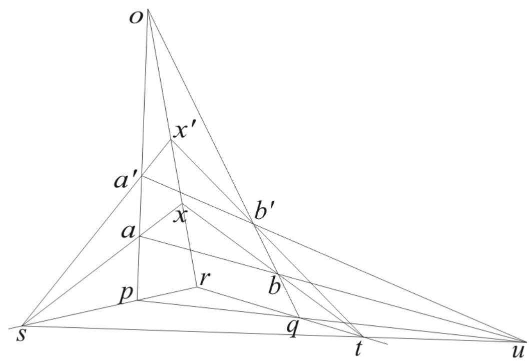

Figure 3.8: Equivalence of pairs

Proposition 3.3.5 *Let be given four distinct points, not in α and none equal to o. Suppose o, a, a' are on a line l and o, b, b' on a line m. Then*

$$f_{a,a'} = f_{b,b'} \Leftrightarrow f_{a,a'}(b) = b' \Leftrightarrow f_{b,b'}(a) = a'$$

Proof. It is trivial that from $f_{a,a'} = f_{b,b'}$ follows $f_{a,a'}(b) = b'$ and $f_{b,b'}(a) = a'$. Conversely, suppose $f_{a,a'}(b) = b'$. We have to show $f_{a,a'}(x) = f_{b,b'}(x)$ for every x. Suppose first $l \neq m$. Let $n = o \vee x$, $p = l \wedge \alpha$, $q = m \wedge \alpha$, $r = n \wedge \alpha$, $\beta = p \vee q \vee r$, see figure 3.8. By hypothesis we have that $a \vee b$ and $a' \vee b'$ meet in a point u on $p \vee q$. Let $\gamma = x \vee a \vee b$. Then γ meets β in a line k. This line contains u. Also it meets $p \vee r$ in a point s and $q \vee r$ in a point t. The plane $k \vee a'$ meets n in a point x'. Now, by construction, we have $f_{a,a'}(x) = f_{b,b'}(x) = x'$. The case $l = m$ is left as an exercise. $\diamond$

Corollary 3.3.6 *If the homologies h_1 and h_2 both leave o and α invariant, and if there is a point $x \neq o$ not in α such that $h_1(x) = h_2(x)$ then $h_1 = h_2$.* $\diamond$

In the construction of the skew field of scalars we will need the following.

Proposition 3.3.7 *Given four different points a, a', b, b' collinear with o, and such that $b' = h_{aa'}(b)$ then*

$$h_{a'b'} = h_{aa'} \circ h_{ab} \circ h_{a'a}$$

Proof. Note that none of the points can be in α nor equal to o. $h_{aa'} \circ h_{ab} \circ h_{a'a}(a') = h_{aa'} \circ h_{ab}(a) = h_{aa'}(b) = b' = h_{a'b'}(a')$. So, by corollary 3.3.6 the two homologies $h_{a'b'}$ and $h_{aa'} \circ h_{ab} \circ h_{a'a}$ are equal. $\diamond$

The homologies h_{ab} and $h_{a'b'}$ are called *conjugate*.

Chapter 4

The Vector Space

Now we have the tools to show that our space $(S, n, \dim, I)$ is isomorphic to the lattice of subspaces of a vector space over some skew field.

We know that each line has at least three points. If they are really that small, we can construct a vector space directly. This will be done in section 4.7. So for the moment we will assume that each line has at least four points. In addition we assume that the dimension of our space is at least 2.

Choose an arbitrary point o and an arbitrary hyperplane $\infty \not\succ o$ - the hyperplane at *infinity* - so $\dim(o) = 0,\ \dim(\infty) = n - 1$.

A *vector* is a point not in ∞, and a *co-vector* is a hyperplane not containing o. The set of vectors resp. co-vectors is denoted by V resp. V^*.

In this chapter we will abbreviate the join of the points x and y to xy.

4.1 Parallel

As soon as we have singled out our hyperplane ∞ we can define 'parallel' lines. Let $L_f = \{x \in S | x \not\preceq \infty,\ \dim(x) = 1\}$ be the collection of 'finite lines', that is: lines not contained in ∞.

Definition 4.1.1 *Two lines $a, b \in L_f$ are* parallel*, notation $a//b$, if either*

- *$a = b$ or*
- *$\mathbf{0} \prec a \wedge b \prec \infty$, that is, if they meet in a point at infinity.*

It is straightforward to prove that this is an equivalence relation on L_f. To each point at infinity p belongs exactly one equivalence class L_p, consisting of the finite lines that contain p.

In general, two lines either coincide, or share a point, or are *skew* (i.e. their meet is $\mathbf{0}$). For finite lines we have: two lines are either parallel (coincident or not), or share a finite point, or are skew.

Definition 4.1.2 *Let be given four distinct vectors a, b, c, d, not on one line, but in one plane α. Let $ab//cd$ and $ad//bc$. Then this configuration of four points and four lines is called the* parallelogram *$abcd$.*

Note that no three points can be collinear. Note also that the parallelograms $abcd$, $bcda$, etc. (8 names) are all equal.

Subsequently one could define a vector - roughly - as an equivalence class of opposite, directed parallelogram-sides. In that case it is not necessary to single out the point o. But since we want to develop our geometry symmetrically (for duality), we have o and ∞ at our disposal, so there is no need for parallelism yet. However, in the development of scalar multiplication (4.3) parallelism will turn out to be useful.

4.2 Addition

We are going to define addition of vectors. The procedure for co-vectors is, of course, exactly dual. This is worked out in my article *Vector spaces and projective geometry* (available from www.mathart.nl).

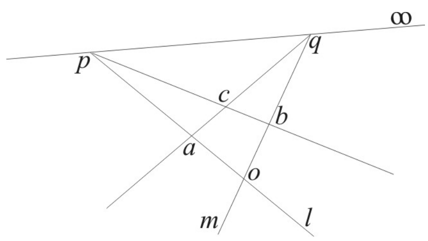

Figure 4.1: The sum of two vectors

Let o, a, b be non-collinear vectors, so $a \not\prec \infty,\ b \not\prec \infty$. By the dimension theorem $l = oa$ is a line and $\alpha = b \vee l$ is a plane. Let $m = ob,\ p = l \wedge \infty$ and $q = m \wedge \infty$, see figure 4.1. Then m is a line and p and q are points. Now pb and qa are different lines in α, hence (dimension theorem) meet in a point c. Suppose $c \prec \infty$, then $qa \prec \infty$ and $pb \prec \infty$ hence $a \prec \infty,\ b \prec \infty$, †. We define the vector $c = a + b := (((o \vee a) \wedge \infty) \vee$

$b) \wedge (((o \vee b) \wedge \infty) \vee a)$. In fact, oc is the diagonal of the parallelogram $oacb$. Because $\wedge$ is commutative we immediately have $a + b = b + a$, in this case.

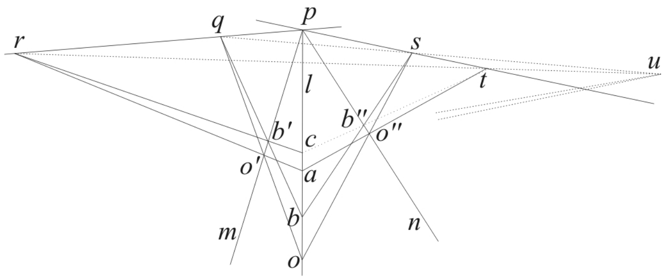

Figure 4.2: The sum of collinear vectors

If o, a, b are on one line l we proceed as follows, see figure 4.2[1]. Let $p = l \wedge \infty$. Because of axiom 2.8.1 there is a point $q \prec \infty$, $q \neq p$. Let $\alpha = l \vee q$, this is a plane. Let $\infty_1 = \alpha \wedge \infty = pq$. There is (corollary 2.9.4) at least one line m in α through p and different from l and ∞_1. Connect q with o, b to get two (not necessarily different) lines, and let the meeting points of these lines with m be o', b' resp. Let $r = \infty_1 \wedge (o'a)$ and $c = l \wedge (b'r)$. Then we define $a + b := c$.

Proposition 4.2.1 *This definition is independent of the choice of q and m.*

Proof. Repeat the construction with n, s in the pane $\beta = l \vee s$, see again figure 4.2. Suppose first that $\alpha \neq \beta$. Note that the whole configuration (exept of course ∞ if the dimension is 4 or more) is in one 3-dimensional subspace, hence two different planes meet in a line, three in a point. The triangles $oo'o''$ and $bb'b''$ are perspective from p hence, by Desargues' theorem, from the line qs. Then $o'o''$, $b'b''$ and qs meet in a point u. The triangles oqs and art are perspective from p hence from a line, which means that $u \prec tr$. Now the triangles art and $pb'b''$ are perspective from a line, hence from c, which means $c \prec tb''$.
If $\alpha = \beta$ then $p,\ q,\ r,\ s,\ t,\ u$ are all on one line ∞_1. The proof, however, is identical. $\diamond$

Verify that this proof holds in the case $a = b$. Prove also that $a + o = o + a = a$, that is, o is the *zero vector*.

[1] As in section 3.3 you should not infer from the figure that we need five points on the line: it is possible that $a = b$ or that c equals one of the other vectors of l; of course $c \neq p$.

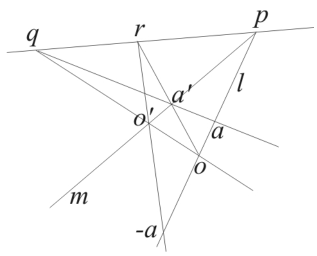

Figure 4.3: The opposite of a vector

The opposite $-a$ of a vector a is defined as in figure 4.3. Again you can prove that this is independent of the choice of m and q. In addition we have - trivially - that $-a + a = o$.

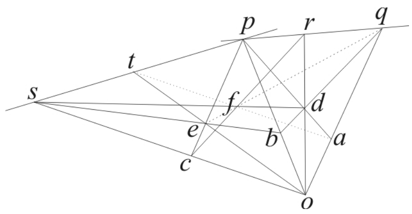

Figure 4.4: Associativity of addition

Proposition 4.2.2 *Addition of vectors is associative.*

Proof. First construct $d = a + b$ in the plane α and $e = b + c$ in β, see figure 4.4. Suppose $\beta \neq \alpha$. Let $f = d + c$, that is $f = ds \wedge cr$. The triangles qdr and esc are perspective from ob, hence from f, hence f is on eq. Next, the triangles est and qda are perspective from op, hence from f: that is $f \prec at$. So, f also equals $a + e$.
If $\alpha = \beta$ then p, q, r, s and t are on one line but the proof is identical. The other cases (2 or 3 vectors collinear with o) are left as exercises. $\diamond$

Proposition 4.2.3 *The pair $(V, +)$ is an abelian group.* $\diamond$

Note that each line through o determines a subgroup, and even: if W is a blade through o then its vectors form a subgroup.

4.3 Scalar multiplication

In the proofs of this section we will restrict to the general cases. The reader is invited to check all other possibilities.

Fix a second vector $e \neq o$, called the *unit vector*. Let k be the set of vectors on oe and remember that k is a subgroup of V. (By abuse of notation we will also denote by k the line oe.) For each $a \in k$ we will define a 'scalar' f_a. As long as $a \neq 0$, this scalar is, in fact, the homology

$$f_a = h_{ea} \ : \ S \ \rightarrow \ S$$

defined by the fixed elements o and ∞ and $f_a(e) = a$, see section 3.3. Since we are dealing with vectors only (and not with lines, planes etc.) we will denote the restriction of f_a to V also by f_a:

$$f_a \ : \ V \ \rightarrow \ V$$

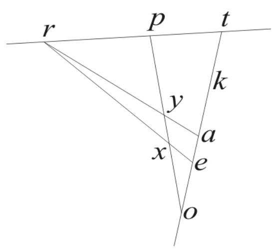

Figure 4.5: Multiplication

For $x \notin k$ we have $f_a(x) = ox \wedge (a \vee (ex \wedge \infty))$, see figure 4.5, where $y = f_a(x)$.

For x on k we have $f_a(x) = a'r \wedge k$, where e' is any vector not in k, $a' = f_a(e')$ and $r = (e'x) \wedge \infty$, see figure 4.6[2], where $y = f_a(x)$. This is independent of the choice of e' by proposition 3.3.1. As a special case we have $f_a(e) = a$.

Now consider the case $a = o$. Obviously the above construction can be done, but it will result in $f_o(x) = o$ for all $x \in V$. Of course f_o is not a homology.

Proposition 4.3.1 *For all $x \in V$ we have*

- $f_e(x) = x$ *, that is:* $f_e = 1_V$*, the identity on* V*,*

[2] Again, if there are less than 6 points on each line several points must coincide, but even then f_a is well-defined.

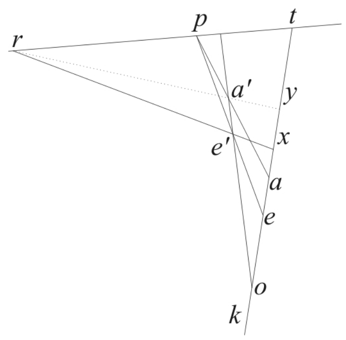

Figure 4.6: Multiplication on k

- $f_o(x) = o$ ⋄

Furthermore: for each blade W through o and each $a \in k, \ a \neq o$, we have $f_a(W) = W$.

A useful proposition is the following.

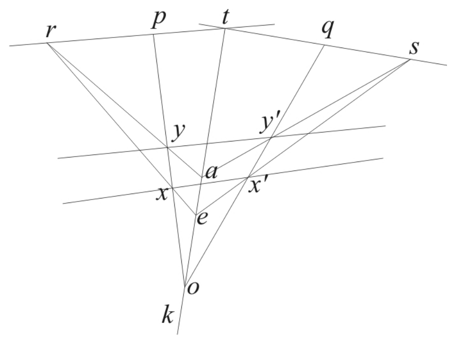

Figure 4.7: Parallel image

Proposition 4.3.2 *For $x \neq x'$, both not on k, $a \in k, \ a \neq 0$ and $y = f_a(x), \ y' = f_a(x')$ we have $yy'//xx'$. Moreover, the vectors $o, \ x, \ y, \ x'$ and y' are in one plane.*

Proof. Let $\alpha = x \vee k$ and $\beta = x' \vee k$, see figure 4.7. Suppose $\alpha \neq \beta$. The triangles exx' and ayy' are perspective from o, hence from pq. So $u = xx' \wedge yy'$ is on $pq \prec \infty$. If $\alpha = \beta$ then $p, \ q, \ r, \ s$ and t are on one line, but the proof is identical. ⋄

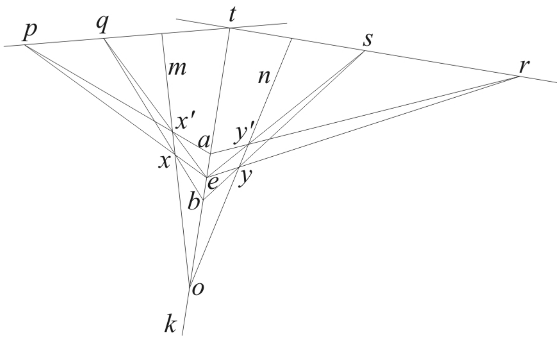

Figure 4.8: Inverse of a scalar

Proposition 4.3.3 *If $a \neq o$ the map f_a is an isomorphism. Moreover, there is a unique $b \in k$ such that $f_b \circ f_a = 1_V$.*

Proof. Since $a \neq o$ our map is a homology, hence an automorphism. Take any line $m \neq k$ through o and any point p in $\infty \wedge (k \vee m)$, but neither on k nor on m, see figure 4.8. Define $x = ep \wedge m$, $x' = ap \wedge m$, $q = ex' \wedge \infty$ and $b = qx \wedge k$. Then b is again independent of the choice of m and p, and f_b is the inverse of f_a. Suppose there is a point c on k with $f_c = f_b$. Then we have $c = f_c(e) = f_b(e) = b$, that is, b is uniquely determined by a. $\diamond$

Note that we have $f_b(a) = f_a(b) = e$ in the above proof.

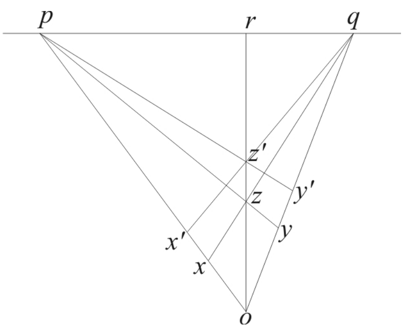

Figure 4.9: Distributivity

Proposition 4.3.4 *Scalar multiplication distributes over vector sums: $f_a(x + y) = f_a(x) + f_a(y)$.*

Proof. In figure 4.9 we have two arbitrary vectors x and y. The lines ox and oy meet ∞ in p, q respectively. First construct $x' = f_a(x)$ and $y' = f_a(y)$. In the plane oxy we construct $z = x+y$ and $z' = f_a(z)$. From proposition 4.3.2 we know that $x'z'//xz$ and $y'z'//yz$. Hence $x'z'//oy'$ and $y'z'//ox'$, which means $z' = x' + y'$. $\diamond$

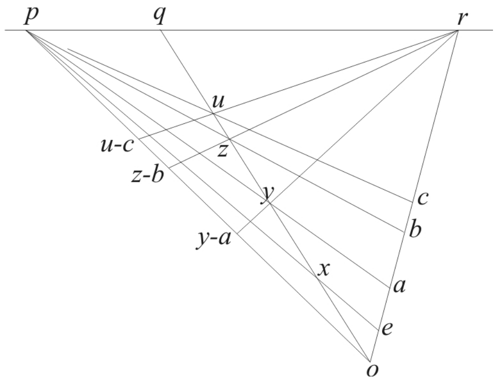

Figure 4.10: Distributivity 2

Proposition 4.3.5 $f_a(x) + f_b(x) = f_{a+b}(x)$

Proof. Let $y = f_a(x),\ z = f_b(x),\ c = a+b,\ u = y+z$, see figure 4.10. We have to prove that $u = f_c(x)$. Note that the lines $ex,\ ay,\ bz$ are parallel, hence share a point $p \prec \infty$. Let $q = ox \wedge \infty,\ r = oe \wedge \infty$. Now project y, z from r on op to get the vectors $y-a,\ z-b$ resp. Because these vectors are collinear with o, so is their sum $(y-a)+(z-b) = (y+z)-(a+b) = u-c$. But $(u-c)+c = u$ hence $cu//op//ex$ which means $u = f_c(x)$. $\diamond$This proof we owe to P. Samuel, see [Samuel] p. 29.

4.4 The skew field of scalars

In the set of scalars, $F = \{f_a | a \in k\}$, we define

- addition by $(f_a + f_b)(x) = f_a(x) + f_b(x)$ and
- multiplication by $f_a f_b = f_a \circ f_b$, composition of maps.

We will show that this makes F a skew field, with $0 = f_o,\ 1 = f_e \neq 0$.

Proposition 4.4.1 *F is closed under addition.*

Proof. By 4.3.5 we have $(f_a + f_b)(x) = f_a(x) + f_b(x) = f_{a+b}(x)$, hence $f_a + f_b = f_{a+b}$ ◇

Proposition 4.4.2 $(F, +)$ *is an additive abelian group.* ◇

In fact it is isomorphic to the additive group k. In particular, addition of scalars is commutative and associative, f_o is the zero-element and $f_{-a} = -f_a$ is the opposite of f_a.

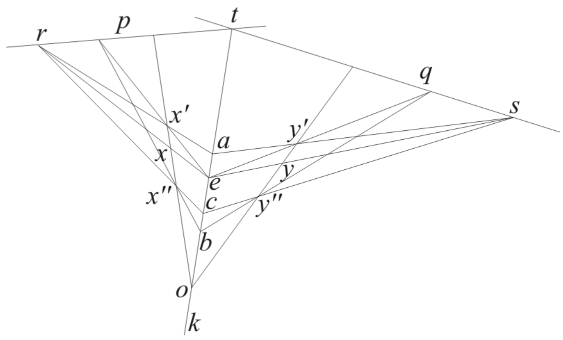

Figure 4.11: Product of scalars

Proposition 4.4.3 *F is closed under multiplication.*

Proof. We have to show that for every $a, b \in k$ there is a $c \in k$ such that $f_b \circ f_a = f_c$. First, define $x' = f_a(x)$ and $x'' = f_b(x')$, see figure 4.11. Next define $c = k \wedge x''r$. As before, one can show that this is independent of x, and that for each y we have $f_a f_b(y) = f_c(y)$. ◇

Note that $c = f_b(a)$ in the above proof.

Proposition 4.4.4 *Multiplication in F is associative.*

Proof. This is a property of map-composition in general. ◇

We also have $f_o f_a(x) = o,\ f_a f_o(x) = f_a(o) = o$ for all x, that is $0f_a = f_a 0 = 0$. Proposition 4.3.3 above stated that if $a \neq o$, that is if $f_a \neq 0$, f_a has an inverse in F. So we have

Proposition 4.4.5 *The non-zero scalars form a group too.* ◇

In particular, f_e is the unit-element.

Proposition 4.4.6 $f_a(f_b + f_c) = f_a f_b + f_a f_c$

Proof. Using 4.3.4 we have $f_a(f_b + f_c)(x) = f_a(f_b(x) + f_c(x)) = f_a f_b(x) + f_a f_c(x)$ $\diamond$

Proposition 4.4.7 $(f_a + f_b)f_c = f_a f_c + f_b f_c$

Proof. $(f_a + f_b)f_c(x) = f_a f_c(x) + f_b f_c(x) = f_p(x) + f_q(x) = (f_p + f_q)(x) = (f_a f_c + f_b f_c)(x)$ $\diamond$

So we found

Proposition 4.4.8 *F is a skew field.* $\diamond$

In addition we have

Proposition 4.4.9 *V is an n-dimensional (left) vector space over F.*

Proof. Veryfy that, indeed, V satisfies the definition of a (left) vector space over F. We will show that its dimension is n. Put $e_1 = e$. Then $V_1 := Fe = k$, a 1-dimensional subspace. So there is an $e_2 \in V \setminus V_1$ and we define $k_2 = e_2 \vee k$ and V_2 is the set of vectors in k_2. By induction we take $e_{j+1} \notin V_j$ and we define $k_{j+1} = e_j \vee k_j$ and V_{j+1} is the set of vectors in k_{j+1}. After a finite number of steps this comes to an end, namely as soon as $k_j = \mathbf{1}$, and then we have the composition chain $\mathbf{0} \prec o \prec k_1 \prec \cdots \prec k_j = 1$. Hence we have $j = n$, so V has a basis $\{e_i\}$. $\diamond$

Now this entire construction of skew field and vector space can be dualized, but this will - of course - give the same set of scalars.

Proposition 4.4.10 *V^* is an n-dimensional (left) vector space over F.* $\diamond$

4.5 Uniqueness of the skew field

In constructing the skew field we have chosen two points and a hyperplane. Do these choices affect the skew field? We will answer this question in several steps.

First, we choose a different 'unity', viz. $e' \in k$ instead of e, which will give us a skew field F'. Note that the elements of both skew fields belong to the group of homologies that leave o and ∞ invariant. We will denote our scalars as homologies now, those of F by f_{ex} and those of F' by $f_{e'x}$.

Obviously there is a bijection $h : F \to F'$ defined by $h(f_{ea}) = f_{e'a'}$ with $a' = f_{ee'}(a)$. We will prove that h is an isomorphism of skew fields, that is, $h(f + g) = h(f) + h(g)$ and $h(fg) = h(f)h(g)$ for all $f, g \in F$.

The first relation is straightforward: $h(f_{ea}) + h(f_{eb}) = f_{e'a'} + f_{e'b'} = f_{e',a'+b'} = f_{e',(a+b)'} = h(f_{e,a+b}) = h(f_{ea} + f_{eb})$, where we used $a' + b' = f_{ee'}(a) + f_{ee'}(b) = f_{ee'}(a+b) = (a+b)'$. The second relation certainly holds if one of the factors vanishes. So suppose $a \neq 0 \neq b$. Now put $f_{ec} = f_{eb} \circ f_{ea}$ with $c = f_{eb}(a)$; $f_{e'c'} = h(f_{ec})$, so $c' = f_{ee'}(c)$; and $f_{e'd'} = f_{e'b'} \circ f_{e'a'}$ with $d' = f_{e'b'}(a')$. Then we have to prove that $f_{e'c'} = f_{e'd'}$ or equivalently $c' = d'$. Or, after substituting $f_{ee'}f_{eb}(a) = f_{e'b'}(f_{ee'}(a))$. But from proposition 3.3.7 we know

$$f_{e'b'} = f_{ee'} \circ f_{eb} \circ f_{e'e}$$

or

$$f_{e'b'} \circ f_{ee'} = f_{ee'} \circ f_{eb}$$

which completes the proof of

Proposition 4.5.1 *Any other unit ($\neq o$) gives an isomorphic skew field.* $\diamond$

Now suppose that we have two different origins o and o', but one hyperplane ∞. Suppose also that the lines k and k' meet in a point $p \prec \infty$. Take any point $z \not\prec \infty$ on the line oo'. Then there is a homology that leaves z and ∞ invariant and which maps o onto o' and k onto k', so the constructed skew fields c.q. vector spaces must be isomorphic too (by the previous proposition it is not necessary that this homology maps one unity onto the other).

If we have one origin and one hyperplane ∞ but different lines k, k', meeting ∞ in p, p' resp. then we take a point z on pp' and a plane α through o but not containing any of the lines k, k'. Now the homology with invariants z and α that maps p onto p' gives an isomorphism of the skew fields constructed with k and k'.

Combining the previous two, we find that as long as there is one fixed hyperplane ∞, all constructed skew fields are isomorphic.

But if we have two different 'hyperplanes at infinity' and one origin we can dualize the whole story, and get an isomorphic skew field once more.

If we have two different origins and two different hyperplanes at infinity, we can find two homologies. The first maps the first origin on the second, leaving one hyperplane at infinity invariant, the second leaves the second origin invariant and maps one hyperplane at infinity onto the other.

Proposition 4.5.2 *For each projective space there is - up to isomorphism - a unique skew field F such that if one hyperplane (resp. one point) is singled out, the remaining set of points (resp. hyperplanes) is an n-dimensional (left-) vector space over F.* $\diamond$

4.6 Pappos' proposition

It is a very remarkable fact that multiplication of scalars is commutative if and only the *Pappos proposition* holds. This links algebra and geometry on a very deep level.

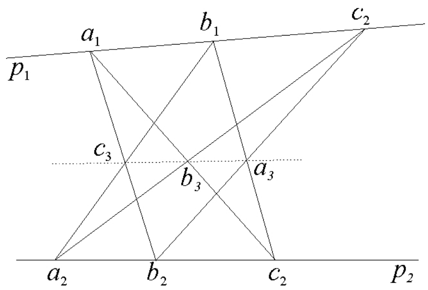

Figure 4.12: Pappos' theorem

Proposition 4.6.1 *of Pappos. For any two distinct coplanar lines p_1 and p_2 and any six distinct points, a_1, b_1, c_1 on p_1, and a_2, b_2, c_2 on p_2, the points $a_3 = (b_1 \vee c_2) \wedge (b_2 \vee c_1)$, $b_3 = (c_1 \vee a_2) \wedge (c_2 \vee a_1)$ and $c_3 = (a_1 \vee b_2) \wedge (a_2 \vee b_1)$ are collinear, see figure 4.12.*

So, this proposition only holds if the skew field is commutative, that is, if it is a field:

Proposition 4.6.2 *Multiplication is commutative if and only if the Pappos proposition holds.*

Proof. Suppose first that the Pappos proposition is true. We have to prove that multiplication is commutative. We already know that $0f_a = f_a0$ and $1f_a = f_a1$. So let $a, b \in k$, both different from 0, 1 and from each other, see figure 4.13. Let $x \notin V \setminus k$ be arbitrary and define $y = f_a(x)$, $z = f_b(x)$ and $v = ar \wedge bq$. Note that the configuration is in a plane $x \vee k$, and that the meet of this plane with ∞ is a line l. Now, on k we have the points e, a, b in that order, and on l we have p, q, r. Connecting these points gives $v = ar \wedge bq$, $eq \wedge ap = y = f_a(x)$ and $er \wedge bp = z = f_b(x)$. Then we have

$$v \prec yz \Leftrightarrow v = f_b f_a(x) = f_a f_b(x) \tag{4.1}$$

But by hypothesis we have $v \prec yz$, hence $f_b f_a(x) = f_a f_b(x)$.
Conversely, suppose multiplication is commutative. Take any pair of distinct lines p_1 and p_2 and any six distinct points, a_1, b_1, c_1 on p_1, and

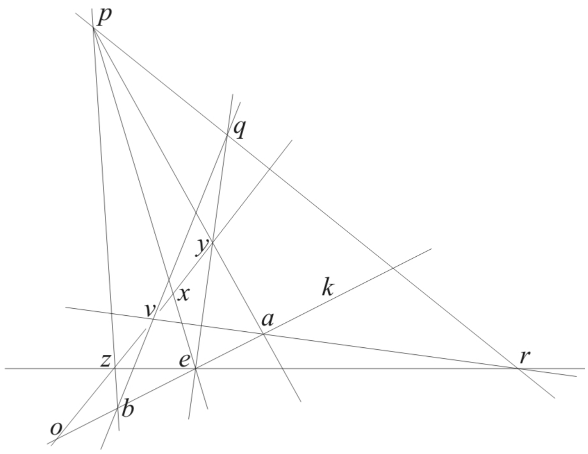

Figure 4.13: Commutativity of multiplication

a_2, b_2, c_2 on p_2. We want to show that the points $a_3,\ b_3$ and c_3 as defined in the proposition, are collinear. If one of the six points is the meeting point of p_1 and p_2 we are done, for then two of the three $a_3,\ b_3,\ c_3$ coincide.
If p_1 and p_2 do meet, but not in one of the six, we proceed as follows. Take any hyperplane $\infty \succeq p_2$ but not containing p_1. Rename our elements to get figure 4.13: $p = a_2,\ q = b_2,\ r = c_2,\ k = p_1,\ c = a_1,\ a = b_1,\ b = c_1$. Then $a_3 = v,\ b_3 = z$ and $c_3 = y$. Define $o = yz \wedge k$. Construct the skew field as before. By hypothesis this skew field is commutative. Hence, by 4.1, v is on yz. $\diamond$

Note: from algebra we know that any finite skew field is commutative, hence, Pappos' theorem holds in finite projective spaces. A geometric proof of this last statement was given by Helga Tecklenburg, see [Tecklenburg].

4.7 Small spaces

We now investigate the case that each line has only three points.

From theorem 2.9.4 we know that all flat pencils have three lines. Now take any plane α and any line l in it and any point $p \prec \alpha$ not on l. Then through p in α we find exactly three lines, meeting l in its three points. By joining points and intersecting lines we can construct at least seven points $p_1 \ldots p_7$ and seven lines $l_1 \ldots l_7$ in α, thus getting a Fano plane inside α. Suppose there is an eighth point q in α. Connect it to any of the seven points by a line m. Then it is immediate that this line must contain another of the p_i, hence coincide with one of the seven lines l_j. And hence q must be one of the seven points p_i. So each plane is a Fano plane. Now take any point o and any hyperplane ∞. Take any plane α through o. This meets ∞ in a line a. The finite points of α are now in 1-1-correspondence with $\mathbf{F}_2^2$. If S is more than 2-dimensional, we can take an arbitrary finite point p_1 outside α and the finite points of $p_1 \vee \alpha$ form a 3-dimensional vector space over $\mathbf{F}_2$. We can repeat this, adding points p_j, until $\alpha \vee p_1 \vee \cdots \vee p_j = \mathbf{1}$.

4.8 The lattice of subspaces

As a last step we will show that each projective space is isomorphic to the lattice of subspaces of a vector space.

So let S be an n-dimensional projective space, with a particular point o and hyperplane ∞. As before we construct a skew field F and a vector space V over F. This vector space has dimension n, and it is isomorphic to F^n. Take any basis $\{e_0, \ldots, e_{n-1}\}$ for V and express every point of V as a set of coordinates with respect to this basis. Embed V in F^{n+1} by $(x_0, \ldots, x_{n-1}) \mapsto (x_0, \ldots, x_{n-1}, 1)$. Let $\mathcal{L}$ be the collection of linear subspaces of F^n Define the map $i : S \to \mathcal{L}$ by

- $i(\mathbf{0}) = \{0\}$
- $i(x_0, \ldots, x_{n-1}) = F \times (x_0, \ldots, x_{n-1}, 1)$ for all vectors $(x_0, \ldots, x_{n-1})$
- if p is a point at infinity, take any vector $(x_0, \ldots, x_{n-1})$ on op and define $i(p) = F \times (x_0, \ldots, x_{n-1}, 0)$
- for any other k-blade $L = p_0 \vee \cdots \vee p_k$ define $i(L) = i(p_0) \oplus \cdots \oplus i(p_k)$.

It is straightforward to prove that i is an isomorphism, the details are left as an exercise for the reader. Thus we have

Proposition 4.8.1 *Every n-dimensional projective space is isomorphic to the lattice of linear subspaces of an $(n+1)$-dimensional vector space.* $\diamond$

4.9 Real geometry

So far we have not required that our intuitive lines (see 1.4) should have infinitely many points, so the skew field could be finite of order p^n with p prime and p^n some very large number. But if that were the case, *separation* of points on a line (see [Coxeter] Chapter II) would not be a projective invariant. Let be given four points A, B, C, D on a line l, see figure 4.14.

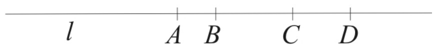

Figure 4.14: separation of points

We say that the points A and C separate the points B and D, which we can denote by $AC||BD$. If we permute the letters we get 12 true and 12 false statements. A projective map may permute the points, but it should respect separation. However, if the (skew) field is finite, separation is *not* invariant (see [Segre] section 121). So, it is only natural to require

- that scalar multiplication is commutative to guarantee that Pappos' proposition holds, and
- that the characteristic of the field is 0, which implies that each line has infinitely many points.

At the same time it is impossible that the field would be $\mathbf{C}$, since the complex numbers are not linearly ordered. In fact, due to considerations about metrics, viz. distances like $\sqrt{2}$ and π, it makes only sense to do 'real' geometry with the real numbers.

So, we now explicitly state that the field is $\mathbf{R}$, thus replacing the axiom of cardinality, 2.9.1, including the dual one.

Axiom 4.9.1 of reality
The skew field of scalars is $\mathbf{R}$.

Corollary 4.9.2 *Every 'real' projective space of dimension* n *is isomorphic to the lattice of linear subspaces of* $\mathbf{R}^{n+1}$. $\diamond$

Corollary 4.9.3 *In these spaces the proposition of Pappos, 4.6.1, holds.* $\diamond$

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List of Symbols

symbol	page	meaning
†	6	contradiction
$\diamond$	6	end of exercise, example, proof; or trivial proof omitted
$\circ$	6	composition of maps
$\wedge$	13	meet
$\vee$	13	join
$\preceq$	9	lies in
$\succeq$	9	contains
$\prec$	10	lies in, but unequal
$\succ$	10	contains, but unequal
S^*	15	the dual of S
$[a, b]$	15	interval
$\uparrow$	12	lies in (in a figure)
$\parallel$	45	parallel
$\mathbf{0}$	11	minimal element of a projective space
$\mathbf{1}$	11	maximal element of a projective space
codim	8	co-dimension
dim	7	dimension
$\mathbf{C}$	6	the field of complex numbers
$\mathbf{F}_p$	6	the field with p elements, p prime
$\mathbf{P}_n$	11	n-dimensional projective space
$\mathbf{Q}$	6	the field of rational numbers
$\mathbf{R}$	6	the field of real numbers
$\mathbf{Z}$	6	the ring of integers
xy	45	the join of the points x and y, in Chapter 4.

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